Homework 3

This homework is due on Sunday, July 10 at 11:59 pm PT

1. Administrivia

(a) The Summer 2022 EECS 16B midterm is on Monday, July 18, 2022, from 7-9 pm PDT. **Please fill out this survey link**, so we can understand your preferences and plan accordingly. To get credit for this problem part, please attach a screenshot indicating that you have completed the survey.

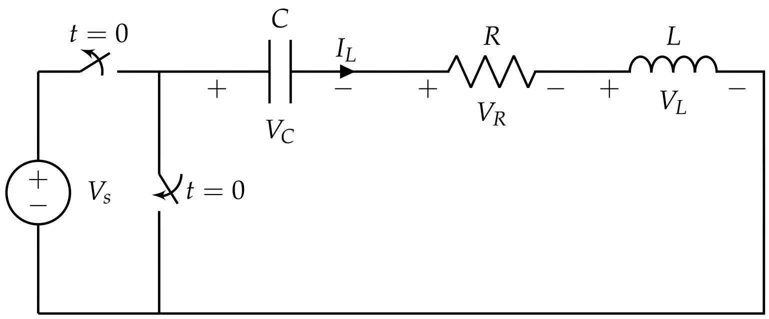
(b) To get credit for this problem part, please attach a screenshot/acknowledgment that you've read this Piazza post about EPA (Effort, Participation, Altruism) Extra Credit. (**Screenshot of your Piazza sidebar without the unread circle for that post is sufficient**)

(c) Homework 1 Grades have been released on Gradescope. Please note that regrades for Homework 1 are due by **Sunday, June 10th, 2022 at 11:59PM**. For your answer to this question, please write an acknowledgment indicating you are aware of the regrade deadline for Homework 1.

2. RLC Responses: Initial Part

Unlocked by Lecture 7

Consider the following circuit:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

The sequence of problems 2 - 5 combined will try to show you the various RLC system responses and how they relate to changing circuit properties.

(a) We first need to construct our state space system. Our state variables are the current through the inductor $x_1(t) = I_L(t)$ and the voltage across the capacitor $x_2(t) = V_C(t)$ since these are the quantities whose derivatives show up in the system of equations governing our circuit. Now, **show that the system of differential equations in terms of our state variables that describes this circuit for $t \geq 0$ is**

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t) \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = \frac{1}{C}x_1(t). \tag{2}$$

(b) **Write the system of equations in vector/matrix form with the vector state variable $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix}$.** This should be in the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$ with a 2×2 matrix A.

(c) **Show that, for the 2×2 matrix A, the two eigenvalues of A are**

$$\lambda_1 = -\frac{1}{2}\frac{R}{L} + \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \tag{3}$$

$$\lambda_2 = -\frac{1}{2}\frac{R}{L} - \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}. \tag{4}$$

(HINT: The quadratic formula will be involved.)

(d) **Under what condition on the circuit parameters R, L, C will A have two distinct real eigenvalues?**

(e) **Under what condition on the circuit parameters R, L, C will A have two imaginary eigenvalues? What will the eigenvalues be in this case?**

(f) Assuming that the circuit parameters are such that there are a pair of (potentially complex) eigenvalues λ_1, λ_2 so that $\lambda_1 \neq \lambda_2$, **show that the corresponding eigenvectors $\vec{v}_{\lambda_1}, \vec{v}_{\lambda_2}$ are**

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1 C} \end{bmatrix} \quad \text{and} \quad \vec{v}_{\lambda_2} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_2 C} \end{bmatrix}. \tag{5}$$

(g) Assuming circuit parameters such that the two eigenvalues of A are distinct, let $V = \begin{bmatrix} \vec{v}_{\lambda_1} & \vec{v}_{\lambda_2} \end{bmatrix}$ be a specific eigenbasis. Consider a coordinate system for which we can write $\vec{x}(t) = V\vec{\widetilde{x}}(t)$. **Show that the $\widetilde{A}$ so that $\frac{\mathrm{d}}{\mathrm{d}t}\vec{\widetilde{x}}(t) = \widetilde{A}\vec{\widetilde{x}}(t)$ is**

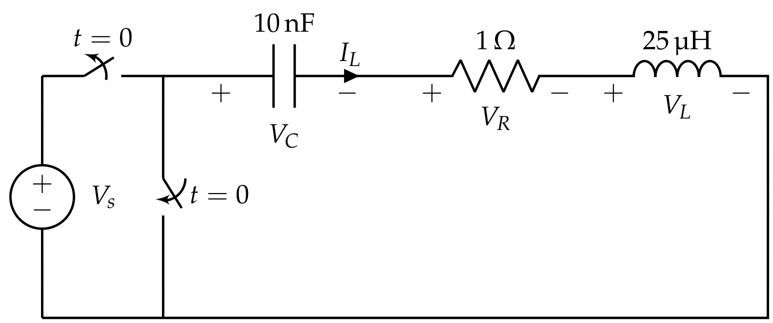
$$\widetilde{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{6}$$

(HINT: Write out the original differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$, then use the given change of coordinates to write everything in terms of $\vec{\widetilde{x}}(t)$.)

3. RLC Responses: Underdamped Case

Unlocked by Lecture 7

Building on the previous problem, consider the following circuit with specified component values:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may round numbers to make the algebra more simple. You may use a calculator or the attached `RLC_Calc.ipynb` Jupyter Notebook for numerical calculations.

(a) Now suppose that $R = 1\,\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1\,\text{V}$. **Find the initial conditions for $\vec{\tilde{x}}(0)$.** Recall that $\vec{\tilde{x}}$ is in the changed "nice" eigenbasis coordinates from the first problem.

(b) Using the diagonalized system from 2(g) and continuing the previous part, **find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.** Remember that your final expressions for $x_1(t)$ and $x_2(t)$ should be real functions (no imaginary terms).

(HINT: Remember that $e^{a+jb} = e^a e^{jb}$. Use Euler's formula.)

(c) In the `RLCSliders.ipynb` Jupyter notebook, move the sliders to approximately $R = 1\Omega$ and $C = 10nF$. **Comment on the graph of $V_C(t)$ and the location of the eigenvalues on the complex plane. Do the waveforms for $x_1(t)$ and $x_2(t)$ decay to 0?**

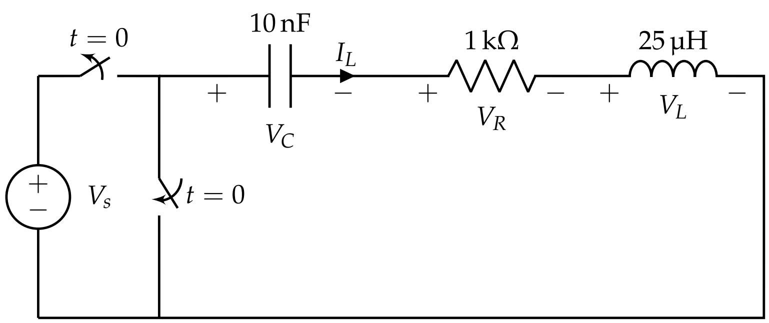
Note: Because the resistance is so small, this is called the "underdamped" case. It is good to reflect upon these waveforms to see why engineers consider such behavior to be reflective of systems that don't have enough damping.

(d) Notice that you got answers in terms of complex exponentials. **Why did the final voltage and current waveforms end up being purely real?**

4. RLC Responses: Overdamped Case

Unlocked by Lecture 7

Building on the previous problem, consider the following circuit with specified component values:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may use a calculator or the attached `RLC_Calc.ipynb` Jupyter Notebook for numerical calculations.

(a) Suppose $R = 1\,\mathrm{k}\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1\,\mathrm{V}$. **Find the initial conditions for $\vec{\tilde{x}}(0)$.** Recall that $\vec{\tilde{x}}$ are the eigenbasis coordinates from the first question.

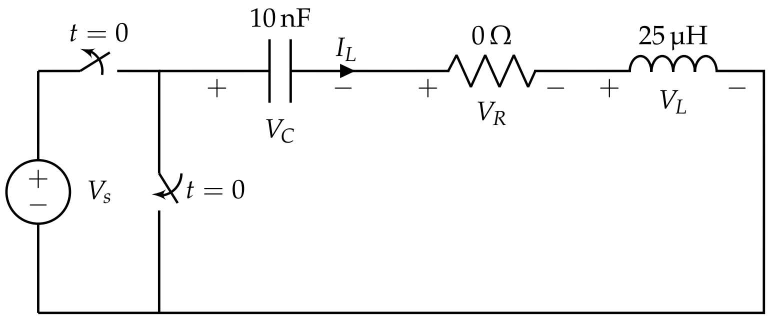
(b) Using the diagonalized system from 2(g) and continuing the previous part, **find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.**

(c) In the `RLCSliders.ipynb` Jupyter notebook, move the sliders to approximately $R = 1\,\mathrm{k}\Omega$ and $C = 10\,\mathrm{nF}$. **Comment on the graph of $V_C(t)$ and the location of the eigenvalues on the complex plane.**

5. (PRACTICE) RLC Responses: Undamped Case

Unlocked by Lecture 7

Building on the previous problem, consider the following circuit with specified component values:



Assume that the capacitor is charged to V_s and there is no current in the inductor for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may use a calculator or the attached `RLC_Calc.ipynb` Jupyter Notebook for numerical calculations.

(a) Suppose $R = 0\,\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1\,\text{V}$. **Find the initial conditions for $\vec{\tilde{x}}(0)$.** Recall that $\vec{\tilde{x}}$ is in the changed "nice" eigenbasis coordinates from the first problem.

(b) Using the diagonalized system from 2(g) and continuing the previous part, **find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.** Remember that your final expressions for $x_1(t)$ and $x_2(t)$ should be real functions (no imaginary terms).

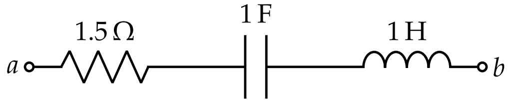
(HINT: Use Euler's formula.)

(c) In the `RLCSliders.ipynb` Jupyter notebook, move the sliders to approximately $R = 0\,\Omega$ and $C = 10\,\text{nF}$. **Comment on the graph of $V_C(t)$ and the location of the eigenvalues on the complex plane. Do the waveforms for $x_1(t)$ and $x_2(t)$ decay to 0?**

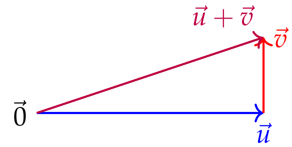
Note: Because there is no resistance, this is called the "undamped" case.

6. Phasors

Unlocked by Lecture 7 and 8



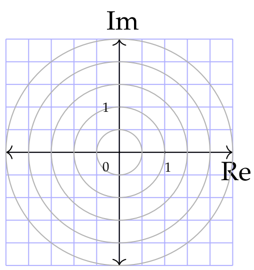
(a) Three components in series.



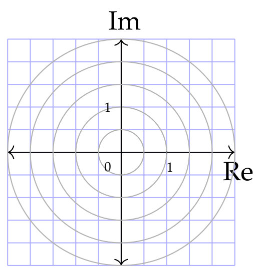
(b) Vector sum of $\vec{u}$ and $\vec{v}$.

Figure 1: Relevant problem figures.

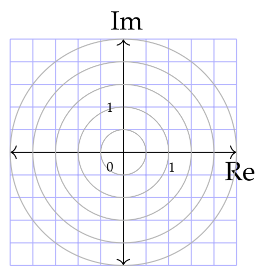
(a) **For the component values given in figure 1a, evaluate the impedances Z_R, Z_C, Z_L and the series equivalent impedance Z_{ab} for the case $\omega = \frac{1}{2}\frac{\text{rad}}{\text{s}}$. Draw the individual impedances as "vectors" on the complex plane. On the last plot draw Z_{ab} as a vector sum (as shown in figure 1b) of Z_R, Z_C, and Z_L on the complex plane. Then give the magnitude and phase of Z_{ab}.**



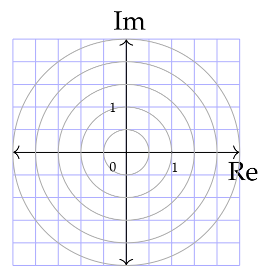
(a) $Z_R(@\omega = 0.5\ \frac{\text{rad}}{\text{s}})$



(b) $Z_C(@\omega = 0.5\ \frac{\text{rad}}{\text{s}})$

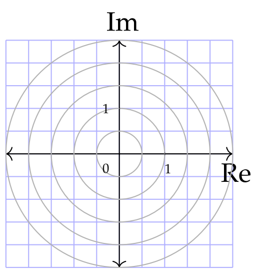


(c) $Z_L(@\omega = 0.5\ \frac{\text{rad}}{\text{s}})$

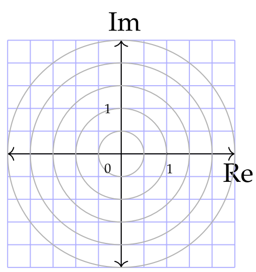


(d) $Z_{\text{ab}}(@\omega = 0.5\ \frac{\text{rad}}{\text{s}})$

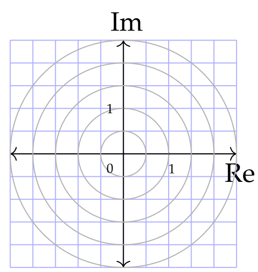
(b) **For the component values given in figure 1a, evaluate the impedances Z_R, Z_C, Z_L and the series equivalent impedance Z_{ab} for the case $\omega = 1\ \frac{\text{rad}}{\text{s}}$. Draw the individual impedances as "vectors" on the complex plane. On the last plot draw Z_{ab} as a vector sum (as shown in figure 1b) of Z_R, Z_C, and Z_L on the complex plane. Then give the magnitude and phase of Z_{ab}.**



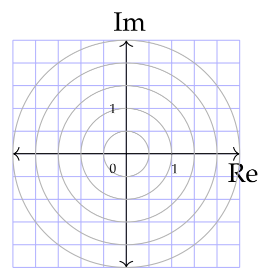
(a) $Z_R(@\omega = 1\ \frac{\text{rad}}{\text{s}})$



(b) $Z_C(@\omega = 1\ \frac{\text{rad}}{\text{s}})$

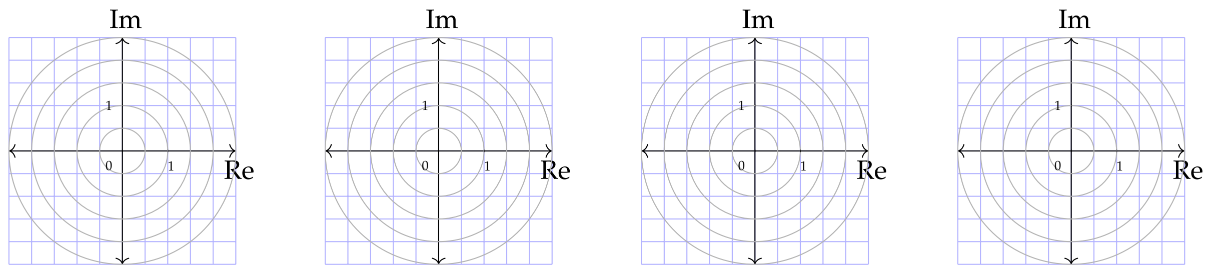


(c) $Z_L(@\omega = 1\ \frac{\text{rad}}{\text{s}})$



(d) $Z_{\text{ab}}(@\omega = 1\ \frac{\text{rad}}{\text{s}})$

(c) **For the component values given in figure 1a, evaluate the impedances Z_R, Z_C, Z_L and the series equivalent impedance Z_{ab} for the case $\omega = 2\ \frac{\text{rad}}{\text{s}}$. Draw the individual impedances as "vectors" on the complex plane. On the last plot draw Z_{ab} as a vector sum (as shown in figure 1b) of Z_R, Z_C, and Z_L on the complex plane. Then give the magnitude and phase of Z_{ab}.**



(a) $Z_R(@\omega = 2\ \frac{\text{rad}}{\text{s}})$ **(b)** $Z_C(@\omega = 2\ \frac{\text{rad}}{\text{s}})$ **(c)** $Z_L(@\omega = 2\ \frac{\text{rad}}{\text{s}})$ **(d)** $Z_{\text{ab}}(@\omega = 2\ \frac{\text{rad}}{\text{s}})$

(d) The "natural frequency" ω_n is defined as the frequency ω_n where the net impendance is purely real. **For the series combination of RLC elements, Z_{ab}, appearing in figure 1a, what is the "natural frequency" ω_n?**

Fact: We call this the "natural frequency" since it is the frequency at which the magnitude of the impedance is the smallest. It turns out to be the case that such a circuit will oscillate at this frequency if it was underdamped (if R was small enough) and we set it up in a problem like that of the underdamped problem on this HW set.

7. Low-pass Filter

Unlocked by Lecture 8 and 9

You have a 1 kΩ resistor and a 1 μF capacitor wired up as a low-pass filter.

(a) **Draw the filter circuit, labeling the input node, output node, and ground.**

(b) **Write down the transfer function of the filter,** $H(j\omega)$ **that relates the output voltage phasor to the input voltage phasor.** Be sure to use the given values for the components.

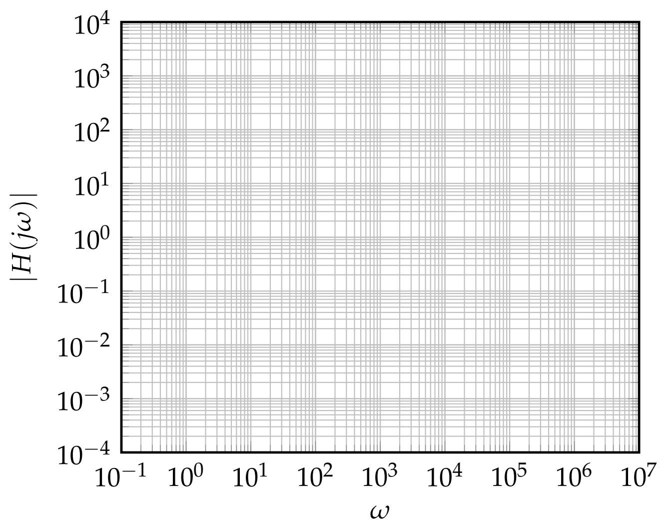
(c) **Write an exact expression for the <u>magnitude</u> of** $H(j\omega = j10^6)$**, and give an approximate numerical answer.**

(d) **Write an exact expression for the <u>phase</u> of** $H(j\omega = j1)$**, and give an approximate numerical answer.**

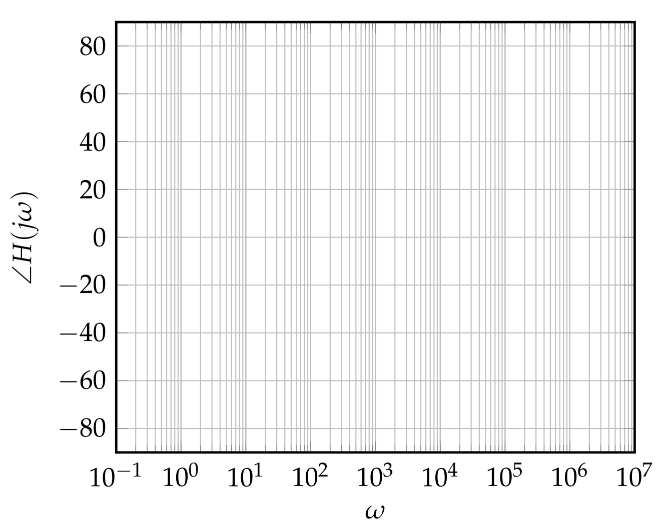
(e) **Write down an expression for the time-domain output waveform** $V_{out}(t)$ **of this filter if the input voltage is** $V(t) = 1\sin(1000t)$ **V.** You can assume that any transients have died out — we are interested in the steady-state waveform.

(f) **Sketch (by hand) the Bode plot (both magnitude and phase) of the filter on the graph paper below.**

Log-log plot of transfer function magnitude



Semi-log plot of transfer function phase



8. Phasors and Eigenvalues

Unlocked by Lecture 7 and 8

Suppose that we have the two-dimensional system of differential equations expressed in matrix/vector form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \tag{7}$$

where for this problem, the matrix A and the vector $\vec{b}$ are both real.

(a) Give a necessary condition on the eigenvalues λ_k of A such that any impact of an initial condition will eventually completely die out. (i.e. the system will reach steady-state.)

You don't have to prove this. The idea here is to make sure that you understand what kind of thing is required. *(HINT: Read Section 2 in Note 5.)*

(b) Now assume that $u(t)$ has a phasor representation $\widetilde{U}$. In other words, $u(t) = \widetilde{U}\mathrm{e}^{+\mathrm{j}\omega t} + \overline{\widetilde{U}}\mathrm{e}^{-\mathrm{j}\omega t}$.

Assume that the vector solution $\vec{x}(t)$ to the system of differential equations (7) can also be written in phasor form as

$$\vec{x}(t) = \widetilde{\vec{X}}\mathrm{e}^{+\mathrm{j}\omega t} + \overline{\widetilde{\vec{X}}}\mathrm{e}^{-\mathrm{j}\omega t}. \tag{8}$$

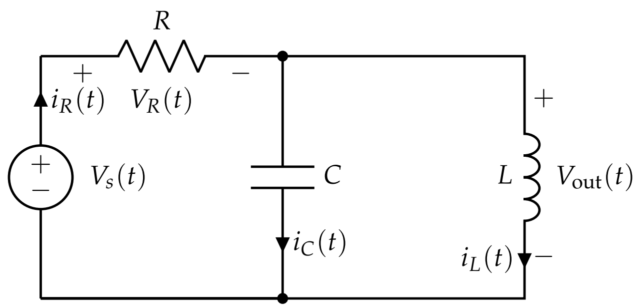
Derive an expression for $\widetilde{\vec{X}}$ involving $A, \vec{b}, \mathrm{j}\omega, \widetilde{U}$, and the identity matrix I. (Here, we assume that $\mathrm{j}\omega$, and $-\mathrm{j}\omega$ are not eigenvalues of A, which indicates that $\det(\mathrm{j}\omega I - A)$ and $\det(-\mathrm{j}\omega I - A)$ are non-zero.)

9. Phasor-Domain Circuit Analysis

Unlocked by Lecture 7 and 8

The analysis techniques you learned previously in 16A for resistive circuits are equally applicable for analyzing circuits driven by sinusoidal inputs in the phasor domain. In this problem, we will walk you through the steps with a concrete example.

Consider the following circuit where the input voltage is sinusoidal. The end goal of our analysis is to find an equation for $V_{\text{out}}(t)$.



The components in this circuit are given by:

$$V_s(t) = 10\sqrt{2}\cos\left(100t - \frac{\pi}{4}\right) \quad (9)$$

$$R = 5\,\Omega \quad (10)$$

$$L = 50\,\text{mH} \quad (11)$$

$$C = 2\,\text{mF} \quad (12)$$

(a) **Give the amplitude V_0, input frequency ω, and phase ϕ of the input voltage V_s.**

(b) Transform the circuit into the phasor domain. **What are the impedances of the resistor, capacitor, and inductor? What is the phasor $\widetilde{V}_S$ of the input voltage $V_s(t)$?**

(c) Use the circuit equations to **solve for** $\widetilde{V}_{\text{out}}$, the phasor representing the output voltage.

(d) **Convert the phasor $\widetilde{V}_{\text{out}}$ back to get the time-domain signal $V_{\text{out}}(t)$.**

10. Homework Process, Study Group, and Course Weekly Survey

Citing sources and collaborators are an important part of life, including being a student!

We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

At the same time, we want to check-in weekly regarding Discussions, Lectures, Lab, and Office Hours and see how effective they have all been for you as students.

Please fill out this survey link. For your submission, please attach a screenshot indicating that you have completed the survey this week.

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- Kris Pister.
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