This homework is due on Sunday, July 10 at 11:59 pm PT

1. Administrivia

(a) The Summer 2022 EECS 16B midterm is on Monday, July 18, 2022, from 7-9 pm PDT. Please fill out this survey link, so we can understand your preferences and plan accordingly. To get credit for this problem part, please attach a screenshot indicating that you have completed the survey.

(b) To get credit for this problem part, please attach a screenshot/acknowledgment that you’ve read this Piazza post about EPA (Effort, Participation, Altruism) Extra Credit. (Screenshot of your Piazza sidebar without the unread circle for that post is sufficient)

(c) Homework 1 Grades have been released on Gradescope. Please note that regrades for Homework 1 are due by Sunday, June 10th, 2022 at 11:59PM. For your answer to this question, please write an acknowledgment indicating you are aware of the regrade deadline for Homework 1.
2. RLC Responses: Initial Part

Unlocked by Lecture 7

Consider the following circuit:

Assume the circuit above has reached steady state for \( t < 0 \). At time \( t = 0 \), the switch changes state and disconnects the voltage source, replacing it with a short.

The sequence of problems 2 - 5 combined will try to show you the various RLC system responses and how they relate to changing circuit properties.

(a) We first need to construct our state space system. Our state variables are the current through the inductor \( x_1(t) = I_L(t) \) and the voltage across the capacitor \( x_2(t) = V_C(t) \) since these are the quantities whose derivatives show up in the system of equations governing our circuit. Now, show that the system of differential equations in terms of our state variables that describes this circuit for \( t \geq 0 \) is

\[
\frac{d}{dt} x_1(t) = -\frac{R}{L} x_1(t) - \frac{1}{L} x_2(t) \tag{1}
\]

\[
\frac{d}{dt} x_2(t) = \frac{1}{C} x_1(t). \tag{2}
\]

Solution: For this part, we need to find two differential equations, each including a derivative of one of the state variables.

First, let’s consider the capacitor equation \( I_C(t) = C \frac{d}{dt} V_C(t) \). In this circuit, \( I_C(t) = I_L(t) \), so we can write

\[
I_C(t) = C \frac{d}{dt} V_C(t) = I_L(t) \tag{3}
\]

\[
\frac{d}{dt} V_C(t) = \frac{1}{C} I_L(t). \tag{4}
\]

If we use the state variable names, we can write this as

\[
\frac{d}{dt} x_2(t) = \frac{1}{C} x_1(t), \tag{5}
\]

so now we have one differential equation.

For the other differential equation, consider the voltage drop across the capacitor, resistor and inductor. At \( t \geq 0 \), the voltage difference between the positive ‘+’ terminal of \( C \) and the negative ‘−’ terminal of \( L \) is given by
\[ V_C(t) + V_R(t) + V_L(t) = 0. \] (6)

Using Ohm’s Law \( V_R(t) = RI_L(t) \) and the inductor equation \( V_L(t) = L \frac{d}{dt} I_L(t) \), we can write this as
\[ V_C(t) + RI_L(t) + L \frac{d}{dt} I_L(t) = 0, \] (7)

which we can rewrite as
\[ \frac{d}{dt} I_L(t) = - \frac{R}{L} I_L(t) - \frac{1}{L} V_C(t). \] (8)

If we use the state variable names, this becomes
\[ \frac{d}{dt} x_1(t) = - \frac{R}{L} x_1(t) - \frac{1}{L} x_2(t), \] (9)

and we have a second differential equation.

To summarize, the final system is
\[ \frac{d}{dt} x_1(t) = - \frac{R}{L} x_1(t) - \frac{1}{L} x_2(t) \] (10)
\[ \frac{d}{dt} x_2(t) = \frac{1}{C} x_1(t). \] (11)

(b) Write the system of equations in vector/matrix form with the vector state variable \( \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \). This should be in the form \( \frac{d}{dt} \vec{x}(t) = A \vec{x}(t) \) with a 2 × 2 matrix \( A \).

Solution: By inspection from the previous part, we have
\[ \begin{bmatrix} \frac{d}{dt} x_1(t) \\ \frac{d}{dt} x_2(t) \end{bmatrix} = \begin{bmatrix} - \frac{R}{L} & - \frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \] (12)

which is in the form \( \frac{d}{dt} \vec{x}(t) = A \vec{x}(t) \), with
\[ A = \begin{bmatrix} - \frac{R}{L} & - \frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}. \] (13)

(c) Show that, for the 2 × 2 matrix \( A \), the two eigenvalues of \( A \) are
\[ \lambda_1 = - \frac{1}{2} \frac{R}{L} + \frac{1}{2} \sqrt{\left( \frac{R}{L} \right)^2 - \frac{4}{LC}} \] (14)
\[ \lambda_2 = - \frac{1}{2} \frac{R}{L} - \frac{1}{2} \sqrt{\left( \frac{R}{L} \right)^2 - \frac{4}{LC}}. \] (15)

(HINT: The quadratic formula will be involved.)

Solution: To find the eigenvalues, we’ll solve \( \det(A - \lambda I) = 0 \). In other words, we want to find \( \lambda \) such that
\[ \det(A - \lambda I) = \det \begin{bmatrix} - \frac{R}{L} - \lambda & - \frac{1}{L} \\ \frac{1}{C} & - \lambda \end{bmatrix} = 0. \] (16)
\[ = -\lambda \left( -\frac{R}{L} - \lambda \right) + \frac{1}{LC} \quad (17) \]
\[ = \lambda^2 + \frac{R}{L} \lambda + \frac{1}{LC} = 0. \quad (18) \]

The quadratic formula gives
\[ \lambda = \frac{-1}{2} \frac{R}{L} \pm \frac{1}{2} \sqrt{\left( \frac{R}{L} \right)^2 - \frac{4}{LC}} \quad (19) \]
as desired.

(d) Under what condition on the circuit parameters \( R, L, C \) will \( A \) have two distinct real eigenvalues?

**Solution:** For both eigenvalues to be real and distinct, we need the quantity inside the square root to be positive. In other words, we need
\[ \frac{R^2}{L^2} - \frac{4}{LC} > 0, \quad (20) \]
or, equivalently,
\[ R > 2\sqrt{\frac{L}{C}}. \quad (21) \]

(e) Under what condition on the circuit parameters \( R, L, C \) will \( A \) have two imaginary eigenvalues? What will the eigenvalues be in this case?  

**Solution:** The only way for both eigenvalues to be purely imaginary is to have \( R = 0 \). In this case, the eigenvalues would be
\[ \lambda = \pm \sqrt{\frac{1}{LC}}. \quad (22) \]

(f) Assuming that the circuit parameters are such that there are a pair of (potentially complex) eigenvalues \( \lambda_1, \lambda_2 \) so that \( \lambda_1 \neq \lambda_2 \), show that the corresponding eigenvectors \( \vec{v}_{\lambda_1}, \vec{v}_{\lambda_2} \) are
\[ \vec{v}_{\lambda_1} = \left[ \begin{array}{c} 1 \\ \frac{1}{\lambda_1 C} \end{array} \right] \quad \text{and} \quad \vec{v}_{\lambda_2} = \left[ \begin{array}{c} 1 \\ \frac{1}{\lambda_2 C} \end{array} \right]. \quad (23) \]

**Solution:** We use the definition of an eigenvector and eigenvalue. We want \( A \vec{v}_{\lambda_i} = \lambda_i \vec{v}_{\lambda_i} \).

Note that, for any \( y \),
\[ A \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} -\frac{y}{L} \\ \frac{y}{C} \end{bmatrix} \quad (24) \]
is not a scalar multiple of \( \begin{bmatrix} 0 \\ y \end{bmatrix} \), so no eigenvector is of the form \( \begin{bmatrix} 0 \\ y \end{bmatrix} \). Thus they must all be of the form \( \begin{bmatrix} y_1 \\ y_2 \\ \frac{1}{y} \end{bmatrix} \) with \( y_1 \neq 0 \), and we can divide through by \( y_1 \) to show that every eigenvector is of the form \( \begin{bmatrix} 1 \\ y_1 \\ \frac{1}{y} \end{bmatrix} \) for some \( y \).

Thus,
\[ \begin{bmatrix} -\frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} = \lambda_i \begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} \lambda_i \\ \lambda_i \cdot y \end{bmatrix} \quad (25) \]
We also know that:

\[
\begin{bmatrix}
-\frac{R}{L} & -\frac{1}{L} \\
\frac{1}{C} & 0
\end{bmatrix}
\begin{bmatrix}
y \\
1
\end{bmatrix}
= 
\begin{bmatrix}
-\frac{R}{L} & -\frac{1}{L} \\
\frac{1}{C} & 0
\end{bmatrix}
\begin{bmatrix}
y \\
1
\end{bmatrix}
\]

(26)

Equating the two equations from above gives:

\[
\begin{bmatrix}
\lambda_i \\
\lambda_i \cdot y
\end{bmatrix}
= 
\begin{bmatrix}
-\frac{R}{L} & -\frac{1}{L} \\
\frac{1}{C} & 0
\end{bmatrix}
\begin{bmatrix}
y \\
1
\end{bmatrix}
\]

(27)

From the second row we see that \( y = \frac{1}{\lambda_i C} \). Now we find the eigenvectors as:

\[
\vec{v}_{\lambda_1} = 
\begin{bmatrix}
1 \\
\frac{1}{\lambda_1 C}
\end{bmatrix}
\]

(28)

\[
\vec{v}_{\lambda_2} = 
\begin{bmatrix}
1 \\
\frac{1}{\lambda_2 C}
\end{bmatrix}
\]

(29)

Alternatively, you can try to use the standard approach of finding the nullspace of \( A - \lambda_i I \) to arrive at the same answer as above.

(g) Assuming circuit parameters such that the two eigenvalues of \( A \) are distinct, let \( V = \begin{bmatrix} \vec{v}_{\lambda_1} & \vec{v}_{\lambda_2} \end{bmatrix} \) be a specific eigenbasis. Consider a coordinate system for which we can write \( \vec{x}(t) = V \vec{\tilde{x}}(t) \).

**Show that the \( \tilde{A} \) so that** \( \frac{d}{dt} \vec{\tilde{x}}(t) = \tilde{A} \vec{\tilde{x}}(t) \) **is**

\[
\tilde{A} = 
\begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}
\]

(30)

(HINT: Write out the original differential equation \( \frac{d}{dt} \vec{x}(t) = A \vec{x}(t) \), then use the given change of coordinates to write everything in terms of \( \vec{\tilde{x}}(t) \).

**Solution:** \( V \) is given by:

\[
V = 
\begin{bmatrix}
1 & 1 \\
\frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C}
\end{bmatrix}
\]

(31)

We know that \( V \) transforms from the \( \vec{\tilde{x}} \) coordinate frame to the \( x \) coordinate frame, \( V^{-1} \) transforms back, and \( A \) takes gives the relationship from \( x \) to \( \frac{d}{dt} \vec{x} \).

Therefore to go from \( \vec{\tilde{x}} \) to \( \frac{d}{dt} \vec{\tilde{x}} \):

\[
\tilde{A} = V^{-1}AV
\]

(32)

\[
= 
\begin{bmatrix}
1 & 1 \\
\frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C}
\end{bmatrix}
\begin{bmatrix}
-\frac{R}{L} & -\frac{1}{L} \\
\frac{1}{C} & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
\frac{1}{\lambda_1 C} \\
\frac{1}{\lambda_2 C}
\end{bmatrix}
\]

(33)

\[
= \frac{\lambda_1 \lambda_2 C}{\lambda_1 - \lambda_2}
\begin{bmatrix}
\frac{1}{\lambda_1 C} & -1 \\
\frac{1}{\lambda_2 C} & 1
\end{bmatrix}
\begin{bmatrix}
\frac{1}{\lambda_1 C} & 0 \\
\frac{1}{\lambda_2 C} & \frac{1}{\lambda_2 C}
\end{bmatrix}
\]

(34)

\[
= \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}
\]

(35)

You didn’t have to multiply things out explicitly. You could have just noticed that the eigenvector matrix will diagonalize the \( A \) matrix such that \( AV = V\Lambda \) or \( V^{-1}AV = \Lambda \), as per one of the problems on the last homework.
3. RLC Responses: Underdamped Case

Unlocked by Lecture 7

Building on the previous problem, consider the following circuit with specified component values:

\[ V_s(t = 0) = 10 \, \text{nF}, \quad V_C(t = 0) = 1 \, \Omega, \quad V_R = 25 \, \mu \text{H}, \quad V_L \]

Assume the circuit above has reached steady state for \( t < 0 \). At time \( t = 0 \), the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may round numbers to make the algebra more simple. You may use a calculator or the attached RLC_Calc.ipynb Jupyter Notebook for numerical calculations.

(a) Now suppose that \( R = 1 \, \Omega \) and the other component values are as specified in the circuit. Assume that \( V_s = 1 \, \text{V} \). Find the initial conditions for \( \tilde{x}(0) \).

Recall that \( \tilde{x} \) is in the changed “nice” eigenbasis coordinates from the first problem. Solution:

Under these conditions, we can solve for

\[
\begin{align*}
\lambda_1 &= -0.02 \times 10^6 + j(2 \times 10^6), \\
\lambda_2 &= -0.02 \times 10^6 - j(2 \times 10^6), \\
V &= \begin{bmatrix} 1 & 1 \\ -0.0002 + j(0.02) & -0.0002 - j(0.02) \end{bmatrix}, \\
V^{-1} &= \begin{bmatrix} 0.5 + j(0.005) & j(0.01) \\ 0.5 - j(0.005) & -j(0.01) \end{bmatrix}, \\
\tilde{x}(0) &= V^{-1}x(0) = V^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} j(0.01) \\ -j(0.01) \end{bmatrix}
\end{align*}
\]

(b) Using the diagonalized system from 2(g) and continuing the previous part, find \( x_1(t) = I_L(t) \) and \( x_2(t) = V_C(t) \) for \( t \geq 0 \). Remember that your final expressions for \( x_1(t) \) and \( x_2(t) \) should be real functions (no imaginary terms).

(HINT: Remember that \( e^{a+jb} = e^a e^{jb} \). Use Euler’s formula.)

Solution:

\[
\tilde{A} = \begin{bmatrix} -0.02 \times 10^6 + j(2 \times 10^6) & 0 \\ 0 & -0.02 \times 10^6 - j(2 \times 10^6) \end{bmatrix}
\]
\[ \begin{bmatrix} \frac{d}{dt} \vec{x}_1(t) \\ \frac{d}{dt} \vec{x}_2(t) \end{bmatrix} = \begin{bmatrix} -0.02 \times 10^6 + j(2 \times 10^6) & 0 \\ 0 & -0.02 \times 10^6 - j(2 \times 10^6) \end{bmatrix} \begin{bmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{bmatrix} \] (42)

Therefore:

\[ \vec{x}_1(t) = K_1 e^{(-0.02 \times 10^6 + j(2 \times 10^6))t} \]
\[ \vec{x}_2(t) = K_2 e^{(-0.02 \times 10^6 - j(2 \times 10^6))t}. \] (43)

Solving for \( K \) with the initial condition gives:

\[ \vec{x}_1(t) = j(0.01) e^{(-0.02 \times 10^6 + j(2 \times 10^6))t} \]
\[ \vec{x}_2(t) = -j(0.01) e^{(-0.02 \times 10^6 - j(2 \times 10^6))t}. \] (44)

Converting back to the \( \vec{x} \) coordinates:

\[ \vec{x}(t) = V \vec{x}(t) = \begin{bmatrix} 1 & 1 \\ -0.5 - j50 & -0.5 + j50 \end{bmatrix} \begin{bmatrix} j(0.01) e^{(-0.02 \times 10^6 + j(2 \times 10^6))t} \\ -j(0.01) e^{(-0.02 \times 10^6 - j(2 \times 10^6))t} \end{bmatrix} \] (47)

\[ x_1(t) = j(0.01) e^{(-0.02 \times 10^6 + j(2 \times 10^6))t} - j(0.01) e^{(-0.02 \times 10^6 - j(2 \times 10^6))t} \]
\[ = j(0.01) e^{-0.02 \times 10^6 t} e^{j(2 \times 10^6)t} - j(0.01) e^{-(0.02 \times 10^6) t} e^{-j(2 \times 10^6)t} \]
\[ = e^{-0.02 \times 10^6 t} \left( j(0.01) e^{j(2 \times 10^6)t} - j(0.01) e^{-j(2 \times 10^6)t} \right) \]
\[ = -0.02 e^{-0.02 \times 10^6 t} \sin \left( 2 \times 10^6 t \right) \] (48)

\[ x_2(t) = (0.5 - j(0.005)) e^{(-0.02 \times 10^6 + j(2 \times 10^6))t} + (0.5 + j(0.005)) e^{(-0.02 \times 10^6 - j(2 \times 10^6))t} \]
\[ = (0.5 - j(0.005)) e^{-0.02 \times 10^6 t} e^{j(2 \times 10^6)t} + (0.5 + j(0.005)) e^{-(0.02 \times 10^6) t} e^{-j(2 \times 10^6)t} \]
\[ = e^{-0.02 \times 10^6 t} \left( (0.5 - j(0.005)) e^{j(2 \times 10^6)t} + (0.5 + j(0.005)) e^{-j(2 \times 10^6)t} \right) \]
\[ = e^{-0.02 \times 10^6 t} \cos \left( 2 \times 10^6 t \right) + 0.01 \cdot e^{-0.02 \times 10^6 t} \sin \left( 2 \times 10^6 t \right). \] (49)

(c) In the RLCSliders.ipynb Jupyter notebook, move the sliders to approximately \( R = 1 \Omega \) and \( C = 10 \mu F \). Comment on the graph of \( V_C(t) \) and the location of the eigenvalues on the complex plane. Do the waveforms for \( x_1(t) \) and \( x_2(t) \) decay to 0? 

Note: Because the resistance is so small, this is called the “underdamped” case. It is good to reflect upon these waveforms to see why engineers consider such behavior to be reflective of systems that don’t have enough damping.

Solution: Yes, the waveforms decay to 0. They appear to be sinusoids that are decaying exponentially. The eigenvalues should be located at coordinates \((-0.02 \times 10^6, 2 \times 10^6)\) and \((-0.02 \times 10^6, -2 \times 10^6)\).
(d) Notice that you got answers in terms of complex exponentials. **Why did the final voltage and current waveforms end up being purely real?**

**Solution:** In this case, it’s because of the complex conjugacy of the quantities in the problem. The eigenvalues and their associated eigenvectors were complex conjugates, as were the transformed solutions $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$. When we applied the inverse transformation to $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$, we added together many complex conjugate terms, and the imaginary parts cancelled out.

Now, was this just a fluke that just happened to line up perfectly? Is there some $A$ matrix out there with real-valued entries that will result in a complex solution? Or is something more profound going on?

It turns to be no fluke. If the entries in the $A$ matrix are real, and the initial condition $\tilde{x}_0$ is real, then the solution to the differential equation $\frac{d}{dt}\tilde{x} = A\tilde{x}$ with $\tilde{x}(0) = \tilde{x}_0$ will also be real, regardless of whether the eigenvalues of $A$ are real, imaginary, or complex. If a matrix $A \in \mathbb{R}^{n \times n}$ has some complex eigenvalues, then those eigenvalues will always arise in complex conjugate pairs. Furthermore, the eigenvectors associated to those eigenvalues arise on complex conjugate pairs. This will lead to the kind of cancellation that you saw in here, every single time.

After all, the quantities that we observe in the world are always purely real, so we would expect that the solutions to our models would also be real-valued.
4. RLC Responses: Overdamped Case

*Unlocked by Lecture 7*

Building on the previous problem, consider the following circuit with specified component values:

![Circuit Diagram]

Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may use a calculator or the attached `RLC_Calc.ipynb` Jupyter Notebook for numerical calculations.

(a) Suppose $R = 1 \text{k}\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1 \text{V}$. Find the initial conditions for $\tilde{x}(0)$. Recall that $\tilde{x}$ are the eigenbasis coordinates from the first question. **Solution:** First, we must find the initial conditions at $t = 0$. Recall that in steady-state, a capacitor acts as an open circuit. Thus, since the circuit is in steady state before $t = 0$, then no current is flowing in the circuit and the entire voltage drop is across the capacitor. Therefore:

\[
\begin{align*}
  x_1(0) &= I_L(0) = 0 \\
  x_2(0) &= V_C(0) = V_s = 1
\end{align*}
\]

Under these conditions, we can use the circuit component values and plug into the expression for the eigenvalues found in Question 2c to find

\[
\lambda_1 = -1.0 \times 10^5, \quad \lambda_2 = -4.0 \times 10^7
\]

Then, using the expression for $V^{-1}$ from Question 2g

\[
V^{-1} = \begin{bmatrix} -0.0025 & -0.001 \\ 1.0025 & 0.001 \end{bmatrix}
\]

and finally

\[
\tilde{x}(0) = V^{-1} \tilde{x}(0) = \begin{bmatrix} -0.0025 & -0.001 \\ 1.0025 & 0.001 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.001 \\ 0.001 \end{bmatrix}
\]

(b) Using the diagonalized system from 2(g) and continuing the previous part, find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$. **Solution:**

Plugging in for the component values gives:
\[
\tilde{A} = \begin{bmatrix}
-1.0 \times 10^5 & 0 \\
0 & -4.0 \times 10^7 
\end{bmatrix}
\]  
(61)

These eigenvalues are the negative reciprocals of the relevant time constants for these modes.

\[
\begin{bmatrix}
\frac{d}{dt}\tilde{x}_1(t) \\
\frac{d}{dt}\tilde{x}_2(t) 
\end{bmatrix} = \begin{bmatrix}
-1.0 \times 10^5 & 0 \\
0 & -4.0 \times 10^7 
\end{bmatrix} \begin{bmatrix}
\tilde{x}_1(t) \\
\tilde{x}_2(t) 
\end{bmatrix},
\]  
(62)

Therefore:

\[
\tilde{x}_1(t) = K_1 e^{-\left(1.0 \times 10^5\right)t} 
\]  
(63)

\[
\tilde{x}_2(t) = K_2 e^{-\left(4.0 \times 10^7\right)t} 
\]  
(64)

Solving for \(K_1\) and \(K_2\) with the initial condition gives:

\[
\tilde{x}_1(t) = -0.001 e^{-\left(1.0 \times 10^5\right)t} 
\]  
(65)

\[
\tilde{x}_2(t) = 0.001 e^{-\left(4.0 \times 10^7\right)t} 
\]  
(66)

Converting back to the \(\vec{x}\) coordinates:

\[
\vec{x}(t) = V \tilde{x}(t) = \begin{bmatrix}
1 & 1 \\
-1000 & -2.5 
\end{bmatrix} \tilde{x}(t) 
\]  
(67)

\[
x_1(t) = -0.001 e^{-\left(1.0 \times 10^5\right)t} + 0.001 e^{-\left(4.0 \times 10^7\right)t} 
\]  
(68)

\[
x_2(t) = e^{-\left(1.0 \times 10^5\right)t} - 0.0025 e^{-\left(4.0 \times 10^7\right)t}. 
\]  
(69)

Note that using the final solution for \(x_2(t)\), we get \(x_2(0) = 0.9975\). But our original initial condition was \(x_2(0) = 1\) in (57). This apparent contradiction is because of rounding errors in the elements of \(V^{-1}\) in (59) and \(V\) in (67). If we use the ‘exact’ values of \(V^{-1}\) and \(V\) (upto machine precision) from \texttt{RLC-Calculator.ipynb} or any other calculator, we will indeed get \(x_2(0) = 1\).

In general, we should always use ‘exact’ values from the calculator for intermediate steps, and round to significant figures only in the final answer. These solutions round the values in the intermediate steps too just for illustrative purposes.

(c) In the \texttt{RLC_Sliders.ipynb} Jupyter notebook, move the sliders to approximately \(\mathit{R} = \mathbf{1}\ k \Omega\) and \(\mathit{C} = 10\ \text{nF}\). \textbf{Comment on the graph of} \(\mathit{V_C}(t)\) \textbf{and the location of the eigenvalues on the complex plane.}

\textbf{Solution:} \(\mathit{V_C}(t)\) looks like a decaying exponential. The eigenvalues lie on the real axis at coordinates \((-1 \times 10^5, 0)\) and \((-4 \times 10^7, 0)\).
Figure 2: $V_C(t)$ and eigenvalues for overdamped case.
5. (PRACTICE) RLC Responses: Undamped Case

Unlocked by Lecture 7

Building on the previous problem, consider the following circuit with specified component values:

![Circuit Diagram]

Assume that the capacitor is charged to $V_s$ and there is no current in the inductor for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may use a calculator or the attached RLC_Calc.ipynb Jupyter Notebook for numerical calculations.

(a) Suppose $R = 0 \, \Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1 \, \text{V}$. **Find the initial conditions for $\tilde{x}(0)$.** Recall that $\tilde{x}$ is in the changed “nice” eigenbasis coordinates from the first problem. **Solution:**

Under these conditions, we can solve for

$$
\lambda = \pm j \sqrt{\frac{1}{LC}} = \pm j \sqrt{\frac{1}{250 \times 10^{-15}}} = \pm j (2 \times 10^6) \implies \lambda_1 = j (2 \times 10^6), \quad \lambda_2 = -j (2 \times 10^6) \quad (70)
$$

Using the rule we derived earlier for finding $V$, we have

$$
V = \begin{bmatrix} 1 & 1 \\ -j50 & j50 \end{bmatrix} \quad (71)
$$

$$
V^{-1} = \begin{bmatrix} 0.5 & j(0.01) \\ 0.5 & -j(0.01) \end{bmatrix} \quad (72)
$$

which lets us say

$$
\tilde{x}(0) = V^{-1} \tilde{x}(0) = \begin{bmatrix} 0.5 & j(0.01) \\ 0.5 & -j(0.01) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} j(0.01) \\ -j(0.01) \end{bmatrix} . \quad (73)
$$

(b) Using the diagonalized system from 2(g) and continuing the previous part, **find** $x_1(t) = I_L(t)$ **and** $x_2(t) = V_C(t)$ for $t \geq 0$. Remember that your final expressions for $x_1(t)$ and $x_2(t)$ should be real functions (no imaginary terms).

(HINT: Use Euler’s formula.)

**Solution:** Plugging in for the component values gives:
\[
\vec{A} = \begin{bmatrix}
 j(2 \times 10^6) & 0 \\
 0 & -j(2 \times 10^6)
\end{bmatrix}
\quad (74)
\]

\[
\begin{bmatrix}
 \frac{d}{dt} \vec{x}_1(t) \\
 \frac{d}{dt} \vec{x}_2(t)
\end{bmatrix} = \begin{bmatrix}
 j(2 \times 10^6) & 0 \\
 0 & -j(2 \times 10^6)
\end{bmatrix} \begin{bmatrix}
 \vec{x}_1(t) \\
 \vec{x}_2(t)
\end{bmatrix}
\quad (75)
\]

Therefore:

\[
\vec{x}_1(t) = K_1 e^{+j(2 \times 10^6)t}
\quad (76)
\]

\[
\vec{x}_2(t) = K_2 e^{-j(2 \times 10^6)t}
\quad (77)
\]

Solving for \(K\) with the initial condition gives:

\[
\vec{x}_1(t) = j(0.01) e^{+j(2 \times 10^6)t}
\quad (78)
\]

\[
\vec{x}_2(t) = -j(0.01) e^{-j(2 \times 10^6)t}
\quad (79)
\]

Converting back to the \(\vec{x}\) coordinates:

\[
\vec{x}(t) = V \vec{x}(t) = \begin{bmatrix}
 1 & 1 \\
 -j50 & j50
\end{bmatrix} \begin{bmatrix}
 j(0.01) e^{+j(2 \times 10^6)t} \\
 -j(0.01) e^{-j(2 \times 10^6)t}
\end{bmatrix}
\quad (80)
\]

\[
x_1(t) = j(0.01) e^{+j(2 \times 10^6)t} - j(0.01) e^{-j(2 \times 10^6)t} = -0.02 \sin((2 \times 10^6)t)
\quad (81)
\]

\[
x_2(t) = 0.5e^{+j(2 \times 10^6)t} + 0.5e^{-j(2 \times 10^6)t} = \cos((2 \times 10^6)t).
\quad (82)
\]

(c) In the RLCSliders.ipynb Jupyter notebook, move the sliders to approximately \(R = 0 \Omega\) and \(C = 10 \text{nF}\). Comment on the graph of \(V_C(t)\) and the location of the eigenvalues on the complex plane. Do the waveforms for \(x_1(t)\) and \(x_2(t)\) decay to 0? Note: Because there is no resistance, this is called the “undamped” case. Solution: No, these waveforms are sinusoids and do not die out over time. The eigenvalues are located on the imaginary axis at coordinates \((0, -2 \times 10^6)\) and \((0, 2 \times 10^6)\).
Note that if the matrix has eigenvalues which are purely imaginary, the resulting system shows oscillations. Although there is no external energy provided to the system after $t > 0$, the system keeps on oscillating forever. The circuit-theoretical explanation of this is that the total energy of the system remains constant as there is no resistor in the circuit which is responsible for power dissipation. At the beginning the capacitor stores the entire energy while the inductor has 0 energy. As $V_C(t)$ starts decreasing, portion of the capacitor energy gets transferred to the inductor. When $V_C(t)$ becomes 0, the entire energy of the capacitor gets transferred to the inductor. This process goes on for infinite time resulting in an oscillatory system. The energy waveforms below show how the energy gets transferred from the capacitor to the inductor and vice-versa.

![Figure 3](image1.png)

**Figure 3:** $V_C(t)$ and eigenvalues for undamped case.

![Figure 4](image2.png)

**Figure 4:** Periodic transfer of stored energy between the capacitor and the inductor.
6. Phasors

Unlocked by Lecture 7 and 8

(a) For the component values given in figure 5a, evaluate the impedances $Z_R$, $Z_C$, $Z_L$ and the series equivalent impedance $Z_{ab}$ for the case $\omega = \frac{1}{2} \text{rad/s}$. Draw the individual impedances as “vectors” on the complex plane. On the last plot draw $Z_{ab}$ as a vector sum (as shown in figure 5b) of $Z_R$, $Z_C$, and $Z_L$ on the complex plane. Then give the magnitude and phase of $Z_{ab}$.

(b) For the component values given in figure 5a, evaluate the impedances $Z_R$, $Z_C$, $Z_L$ and the series equivalent impedance $Z_{ab}$ for the case $\omega = 1 \text{rad/s}$. Draw the individual impedances as “vectors” on the complex plane. On the last plot draw $Z_{ab}$ as a vector sum (as shown in figure 5b) of $Z_R$, $Z_C$, and $Z_L$ on the complex plane. Then give the magnitude and phase of $Z_{ab}$.

Solution:

Substituting for $\omega = \frac{1}{2} \text{rad/s}$ in the impedance formulas, we get: $Z_R = 1.5 \Omega$, $Z_C = -j2 \Omega$ and $Z_L = j0.5 \Omega$. Since the elements are in series, $Z_{ab} = Z_L + Z_C + Z_R = (1.5 - j1.5) \Omega$. $Z_{ab}$ has magnitude $\sqrt{(1.5 \Omega)^2 + (-1.5 \Omega)^2} = 1.5\sqrt{1 + 1} \Omega = 1.5\sqrt{2} \Omega$ and phase $\text{atan2}(-1.5, 1.5) = -\frac{\pi}{4}$ rad or $-45^\circ$. Following are the plots:

... (plots of impedances and vector sums)
Solution:

Following the same method as last time, with \( \omega = 1 \text{ rad/s} \), we can compute \( Z_R = 1.5 \Omega \), \( Z_C = -j0.5 \Omega \), \( Z_L = j2 \Omega \), and \( Z_{ab} = 1.5 \Omega \). \( Z_{ab} \) has magnitude 1.5 \( \Omega \) and phase 0 rad or 0\(^\circ\).

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(d) The “natural frequency” $\omega_n$ is defined as the frequency $\omega_n$ where the net impedance is purely real. For the series combination of RLC elements, $Z_{\text{ab}}$, appearing in figure 5a, what is the “natural frequency” $\omega_n$?

Fact: We call this the “natural frequency” since it is the frequency at which the magnitude of the impedance is the smallest. It turns out to be the case that such a circuit will oscillate at this frequency if it was underdamped (if R was small enough) and we set it up in a problem like that of the underdamped problem on this HW set.

Solution:

From our above answers, the natural frequency, $\omega_n = 1 \frac{\text{rad}}{s}$. This is where the imaginary parts of the impedance cancel each other.
7. Low-pass Filter

*Unlocked by Lecture 8 and 9*

You have a 1 kΩ resistor and a 1 µF capacitor wired up as a low-pass filter.

(a) **Draw the filter circuit, labeling the input node, output node, and ground.**

   **Solution:**

   ![Figure 12: A simple RC circuit](image)

(b) **Write down the transfer function of the filter, \( H(j\omega) \) that relates the output voltage phasor to the input voltage phasor.** Be sure to use the given values for the components.

   **Solution:** First, we convert everything into the phasor domain. We have,

   \[
   Z_R = R = 1 \times 10^3 \Omega
   \]
   \[
   Z_C = \frac{1}{j\omega C} = \frac{1}{j\omega \times 10^{-6}} F
   \]

   In phasor domain, we can treat these impedances essentially like we treat resistors and recognize the voltage divider. Hence,

   \[
   \tilde{V}_{out} = \frac{Z_C}{Z_C + Z_R} \tilde{V}_{in}
   \]
   \[
   \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = H(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}
   \]
   \[
   = \frac{1}{1 + j\omega RC}
   \]
   \[
   = \frac{1}{1 + j\omega \times 10^{-3}}
   \]

   For any system, a corner/cutoff frequency is the point in the system’s frequency response after which the input begins to be attenuated. More concretely, suppose that for a particular system the corner frequency is \( \omega_C \). Then this means that for input signals with frequency \( \omega_{in} << \omega_C \) the gain of the system is approximately 1 and for input signals with frequency \( \omega_{in} >> \omega_C \) the gain tends to 0. For our particular circuit, the corner frequency \( \omega_C = \frac{1}{RC} = \frac{1}{10^3 \times 10^{-6}} = 10^3 \) rad/sec.

   (c) **Write an exact expression for the magnitude of \( H(j\omega = j10^6) \), and give an approximate numerical answer.**
Solution: From the previous subpart

\[ H(j\omega) = \frac{1}{1 + j\omega/\omega_C}. \]

Then

\[ |H(j\omega)| = \frac{|1|}{|1 + j\omega/\omega_C|}. \]

Since \(|1 + j\omega/\omega_C| = \sqrt{1 + \omega^2/\omega_C^2}\),

\[ |H(j\omega)| = \frac{1}{\sqrt{1 + \omega^2/\omega_C^2}}. \]

Plugging in for \(\omega = 10^6\):

\[ |H(j\omega = 10^6)| = \frac{1}{\sqrt{1 + 10^{12}}}. \]

Approximately:

\[ |H(j\omega = 10^6)| \approx 10^{-3}. \]

(d) Write an exact expression for the phase of \(H(j\omega = j1)\), and give an approximate numerical answer.

Solution: As before

\[ H(j\omega) = \frac{1}{1 + j\omega/\omega_C}. \]

Then

\[ \angle H(j\omega) = \angle 1 - \angle(1 + j\omega/\omega_C) = -\angle(1 + j\omega/\omega_C) = \angle(1 - j\omega/\omega_C). \]

Thus the expression for the transfer function’s phase is given by:

\[ \angle H(j\omega) = \text{atan2}(\frac{\omega}{\omega_C}, 1) \]

Plugging in for \(\omega = 1\):

\[ \angle H(j\omega = 1) = \text{atan2}(\frac{10^6}{10^3}, 1) = \tan^{-1}(-10^{-3}) \]

By the small angle approximation, this is:

\[ \angle H(j\omega = 1) \approx -10^{-3} \text{ rad} \]

(e) Write down an expression for the time-domain output waveform \(V_{out}(t)\) of this filter if the input voltage is \(V(t) = 1 \sin(1000t)\) V. You can assume that any transients have died out — we are interested in the steady-state waveform.

Solution: We can find the magnitude and phase of the transfer function at this point:

\[ |H(j\omega = 10^3)| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \]
\[ \angle H(j\omega = 10^3) = \text{atan2}(-1, 1) = -45^\circ \]

Therefore the output will be:

\[ V_{out}(t) = \frac{1}{\sqrt{2}} \sin(1000t - 45^\circ) \]

(f) Sketch (by hand) the Bode plot (both magnitude and phase) of the filter on the graph paper below.

![Log-log plot of transfer function magnitude](image1)

![Semi-log plot of transfer function phase](image2)

**Solution:**

![Log-log plot of transfer function magnitude](image3)

![Semi-log plot of transfer function phase](image4)
8. Phasors and Eigenvalues

Unlocked by Lecture 7 and 8

Suppose that we have the two-dimensional system of differential equations expressed in matrix/vector form:

$$\frac{d}{dt} \vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)$$  \hspace{1cm} (91)

where for this problem, the matrix $A$ and the vector $\vec{b}$ are both real.

(a) Give a necessary condition on the eigenvalues $\lambda_k$ of $A$ such that any impact of an initial condition will eventually completely die out. (i.e. the system will reach steady-state.)

You don’t have to prove this. The idea here is to make sure that you understand what kind of thing is required. (HINT: Read Section 2 in Note 5.)

Solution: (Recall how the diagonalization we have done in the past takes us to an coordinate system where the matrix representing the differential equation has only diagonal entries being the eigenvalues, corresponding to differential equations of the form $\frac{d}{dt} z(t) = \lambda z(t)$.)

The condition is that all eigenvalues must have real parts that are less than zero. In equations

$$\forall k, \text{Re}(\lambda_k) < 0$$  \hspace{1cm} (92)

This condition derives from the fact that the solutions to differential equations in the eigenspace contain terms that look like $e^{\lambda t}$. So, if all the eigenvalues are have strictly negative real parts, then all such exponential terms will die out.

If any of the eigenvalues have strictly positive real parts, then the exponential terms corresponding to them will blow up as growing exponentials.

The case of $\lambda = 0$ or having a zero real part in general (purely imaginary eigenvalues) is a little more ambiguous in feeling. This suggests that some constant offset (for the case of $\lambda = 0$) or some steady oscillation at a natural frequency of the system can persist throughout all time. But persisting isn’t dying out and so we really want the eigenvalues to have strictly negative real parts for us to be able to ignore the initial conditions.

The argument above implicitly assumes that we can find enough linearly independent eigenvectors to get a basis. But what if we can’t? We will explicitly address that case later in the course, but so far, we have seen in the cases that we have explored that what seems to happen is that even in the new basis, we seem to get a copy of an existing eigenvalue showing up again. This gives us some confidence that the condition that we are expressing is probably the right one, but we aren’t fully sure yet since we have no proof that covers not having enough eigenvectors.

We also know that these kinds of “not enough eigenvectors” cases can occur in physical circuits, since we saw the critically damped case in a previous homework.

(b) Now assume that $u(t)$ has a phasor representation $\bar{U}$. In other words, $u(t) = \bar{U}e^{j\omega t} + \bar{U}e^{-j\omega t}$.

Assume that the vector solution $\vec{x}(t)$ to the system of differential equations (91) can also be written in phasor form as

$$\vec{x}(t) = \bar{X}e^{j\omega t} + \bar{X}e^{-j\omega t}.$$  \hspace{1cm} (93)
Derive an expression for $\tilde{X}$ involving $A, \tilde{b}, j\omega, \tilde{U}$, and the identity matrix $I$. (Here, we assume that $j\omega$, and $-j\omega$ are not eigenvalues of $A$, which indicates that $\det(j\omega I - A)$ and $\det(-j\omega I - A)$ are non-zero.)

Solution: As the hint suggests, plugging back (93) into (91) we get the following:

$$\frac{d}{dt}(\tilde{X}e^{j\omega t} + \tilde{X}e^{-j\omega t}) = A(\tilde{X}e^{j\omega t} + \tilde{X}e^{-j\omega t}) + \tilde{b}(\tilde{U}e^{j\omega t} + \tilde{U}e^{-j\omega t}) \tag{94}$$

$$ (j\omega \tilde{X}e^{j\omega t} - j\omega \tilde{X}e^{-j\omega t}) = (A\tilde{X} + \tilde{b}\tilde{U})e^{j\omega t} + (A\tilde{X} + \tilde{b}\tilde{U})e^{-j\omega t} \tag{95}$$

Note that $\tilde{X}$ and $\tilde{U}$ do not depend on time since they are phasors. Next, we can group the coefficients with the same exponential terms,

$$j\omega \tilde{X} = A\tilde{X} + \tilde{b}\tilde{U} \tag{97}$$

$$ -j\omega \tilde{X} = A\tilde{X} + \tilde{b}\tilde{U} \tag{98}$$

$$\Rightarrow (j\omega)\tilde{X} = (A\tilde{X} + \tilde{b}\tilde{U}) \tag{99}$$

$$\Rightarrow (j\omega)\tilde{X} = (A\tilde{X} + \tilde{b}\tilde{U}) \tag{100}$$

We see that equations (97) and (100) match, which is good. Note that, here we are assuming $A$ and $\tilde{b}$ are real. Next, we can solve (97) to get $\tilde{X}$:

$$j\omega \tilde{X} = A\tilde{X} + \tilde{b}\tilde{U} \tag{101}$$

$$\Rightarrow (j\omega - A)\tilde{X} = \tilde{b}\tilde{U} \tag{102}$$

$$\Rightarrow \tilde{X} = (j\omega - A)^{-1}\tilde{b}\tilde{U}. \tag{103}$$

Notice that we didn’t need to explicitly deal with the conjugate terms. We know that their solution is just going to be the conjugate of what we computed here, because of the properties of complex arithmetic.

It turns out that it is possible to invert a general matrix $M$ by writing it as some matrix $M_c$ (that depends on $M$) divided by the determinant of $M$. (This is a fact related to something called the adjoints of matrices that are studied when one considers a combinatorial perspective on determinants, and thinks about things that are sometimes called “cofactors”.) This is not something that is covered in 16AB because it cannot be proved at the level of mathematical maturity that is fair to assume for courses at this level.

However, the above linear-algebraic fact has a consequence for transfer functions. It tells you that all the polynomial terms in the denominators of the transfer functions are going to have the eigenvalues of the system as their roots. Why? Because the roots of $\det(j\omega I - A)$ tell you the eigenvalues of $A$. In later courses like 120, 105, 140, and beyond, you will see these roots of the denominators referred to as “poles” based on terminology from complex analysis. When you see them, understand that they are just the eigenvalues of the system in disguise. When you see conversations in later courses (or in your job or research) about understanding the placement of poles, understand that what is being talked about is where the relevant eigenvalues of the system are.
9. Phasor-Domain Circuit Analysis

Unlocked by Lecture 7 and 8

The analysis techniques you learned previously in 16A for resistive circuits are equally applicable for analyzing circuits driven by sinusoidal inputs in the phasor domain. In this problem, we will walk you through the steps with a concrete example.

Consider the following circuit where the input voltage is sinusoidal. The end goal of our analysis is to find an equation for $V_{\text{out}}(t)$.

![Circuit Diagram]

The components in this circuit are given by:

\[ V_s(t) = 10\sqrt{2} \cos \left(100t - \frac{\pi}{4}\right) \]  \hspace{1cm} (104)
\[ R = 5 \, \Omega \]  \hspace{1cm} (105)
\[ L = 50 \, \text{mH} \]  \hspace{1cm} (106)
\[ C = 2 \, \text{mF} \]  \hspace{1cm} (107)

(a) Give the amplitude $V_0$, input frequency $\omega$, and phase $\phi$ of the input voltage $V_s$.

Solution: A sinusoid takes the form $v(t) = V_0 \cos(\omega t + \phi)$. Given $V_s(t)$, we find:

\[ V_0 = 10\sqrt{2} \, \text{volt} \]  \hspace{1cm} (108)
\[ \omega = 100 \, \text{rad/sec} \]  \hspace{1cm} (109)
\[ \phi = -\frac{\pi}{4} \, \text{rad} \]  \hspace{1cm} (110)

(b) Transform the circuit into the phasor domain. What are the impedances of the resistor, capacitor, and inductor? What is the phasor $\bar{V}_s$ of the input voltage $V_s(t)$?

Solution:

\[ Z_L = j\omega L = j5\Omega \]  \hspace{1cm} (111)
\[ Z_C = \frac{1}{j\omega C} = -j5\Omega \]  \hspace{1cm} (112)
\[ Z_R = R = 5\Omega \]  \hspace{1cm} (113)
\[ \bar{V}_s = \frac{|V_s|}{2} e^{j\omega t} = 5\sqrt{2}e^{-j\pi/4} \]  \hspace{1cm} (114)
(c) Use the circuit equations to solve for $\tilde{V}_{\text{out}}$, the phasor representing the output voltage.

**Solution:** The phasor representation of the circuit is shown below:

![Circuit Diagram]

Where

$$\tilde{I}_R = \frac{\tilde{V}_S - \tilde{V}_{\text{out}}}{R} \quad (115)$$

$$\tilde{I}_L = \frac{\tilde{V}_{\text{out}}}{j\omega L} \quad (116)$$

$$\tilde{I}_C = \tilde{V}_{\text{out}} \cdot j\omega C \quad (117)$$

Rewriting the current relation in terms of voltage phasors gives:

$$\frac{\tilde{V}_S - \tilde{V}_{\text{out}}}{R} = \tilde{V}_{\text{out}} + j\omega C \tilde{V}_{\text{out}} \quad (118)$$

Substituting the component values in the above equation we get

$$\frac{\tilde{V}_S - \tilde{V}_{\text{out}}}{5} = \frac{\tilde{V}_{\text{out}}}{5j} + \frac{\tilde{V}_{\text{out}} \cdot j}{5} \quad (119)$$

$$= \frac{\tilde{V}_{\text{out}}}{5j} - \frac{\tilde{V}_{\text{out}}}{5j} \quad (120)$$

$$= 0 \quad (121)$$

Which gives:

$$\tilde{V}_{\text{out}} = \tilde{V}_S \quad (122)$$

We found that $\tilde{V}_{\text{out}} = \tilde{V}_S$ because this circuit is in resonance; i.e., the capacitor and inductor have the exact values that cause current and voltage to endlessly oscillate between them at this frequency. If we chose a different value for $\omega$ with these same component values, the circuit would not be in resonance and $\tilde{V}_{\text{out}}$ and $\tilde{V}_S$ would no longer be equal.

One may think that this answer seems weird. For $\tilde{V}_{\text{out}}$ to equal $\tilde{V}_S$ means that no current is flowing through the resistor. This means that somehow, the impedance of the parallel $L$ and $C$ combination would have to be infinity. Let’s check what that is:

$$Z_L \parallel Z_C = \frac{(j5) \cdot (-j5)}{j5 + (-j5)} = +\infty \quad (123)$$
Wow! Indeed it is infinity. This shows something counterintuitive that can occur with phasors and impedances. For resistors, one may think that parallel connections always lower the resistance. However, since imaginary impedances can be positive imaginary and negative imaginary, a parallel connection can make the impedance bigger or smaller. The same kind of counterintuitive behavior is also possible for series combinations. Resistors in series always increase the resistance. But the same L and C in series can combine to have a zero impedance at the natural frequency.

If one wants to know why something divided by 0 is $\infty$ in the complex plane, read this Wiki article: Riemann Sphere. This is another facet of complex analysis, and why engineers were drawn to it when modeling physical systems for design purposes.

(d) Convert the phasor $V_{\text{out}}$ back to get the time-domain signal $V_{\text{out}}(t)$.

**Solution:** Since $V_{\text{out}} = \tilde{V}_S$, 

$$v_{\text{out}}(t) = 10\sqrt{2} \cos\left(100t - \frac{\pi}{4}\right)$$

(124)
10. Homework Process, Study Group, and Course Weekly Survey

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking,
so we can change things in the future if possible.

At the same time, we want to check-in weekly regarding Discussions, Lectures, Lab, and Office Hours
and see how effective they have all been for you as students.

Please fill out this survey link. For your submission, please attach a screenshot indicating that you
have completed the survey this week.

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