Homework 4

This homework is due on Sunday, July 17 at 11:59 pm PT.

1. Administrivia

(a) If you have opted into the remote version of the midterm exam, we will ask you to do a dry run of the remote proctoring guidelines using Homework 4 as the practice assignment. On Wednesday, July 13, you will be emailed a formal list of instructions and a Zoom room for your proctoring dry run. Please proctor yourself doing one written problem of the homework assignment and finally submitting the assignment on Gradescope in one sit-through.

There will be a few differences between the dry run and the actual exam:

- On the day of the exam, we will email you the exam from the eecs16b-su22@berkeley.edu account.
- On the day of the exam, you will email your completed exam to eecs16b-su22@berkeley.edu instead of submitting to Gradescope.
- On the day of the exam, we will not allow you to have EECS 16B resources open on your computer, but while doing homework you may utilize all the appropriate resources you need.

To get credit for the problem, please acknowledge that you are either taking the exam in-person or that are taking the exam remotely and have completed the exam proctoring dry run.

(b) As we approach the second half of the course, we would like to give you all the opportunity to be a part of a lab group. Please fill out this google form if you would like to be placed in a lab group.

To get credit for the problem, please acknowledge that you would not like to be placed in a lab group or attach a screenshot indicating that you have completed the google form.

2. Bandpass Filter: Lowpass and Highpass Cascade

Unlocked by Lecture 10

In lecture, you heard about how you can go through the design of a bandpass filter by cascading lowpass and highpass filters via buffers (op-amps in unity-gain negative feedback to prevent loading effects). In this problem, you will do this for yourself.

Consider an input signal that is composed of the superposition of:

- $A_p := 20\,\text{mV}$ level pure tone at frequency $f_p := 60\,\text{Hz}$ and phase ϕ_p corresponding to power line noise.
- $A_v := 1\,\text{mV}$ level pure tone at frequency $f_v := 600\,\text{Hz}$ and phase ϕ_v corresponding to a voice signal.
- $A_f := 10\,\text{mV}$ level pure tone at frequency $f_f := 60\,\text{kHz}$ and phase ϕ_f corresponding to fluorescent light control electronics noise.

We would like to keep the 600 Hz tone, which could correspond to a voice signal.

NOTE: The phases ϕ are symbolic – we do not provide numerical values – but the amplitudes A are not symbolic.

(a) **Write the $V_{\text{in}}(t)$ that describes the above input in time domain, in the following format:**.

$$V_{\text{in}}(t) = A_p \cos(2\pi f_p t + \phi_p) + A_v \cos(2\pi f_v t + \phi_v) + A_f \cos\left(2\pi f_f t + \phi_f\right) \tag{1}$$

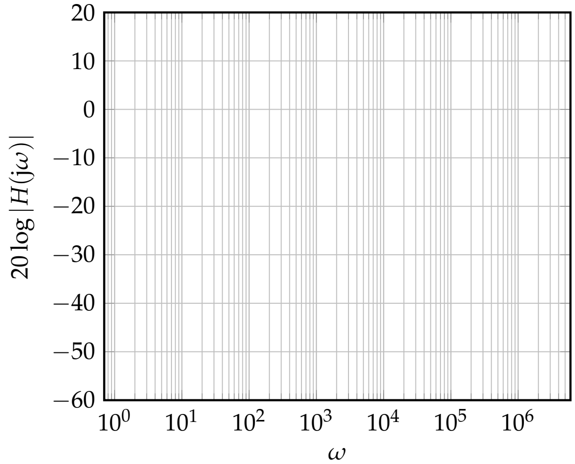
(b) **What are the angular frequencies (i.e., $\omega_p, \omega_v, \omega_f$) involved and the phasors associated with each tone?** Remember that the frequencies of the tones are provided in Hz. To convert these frequencies to angular frequencies, we use $\omega = 2\pi f$.

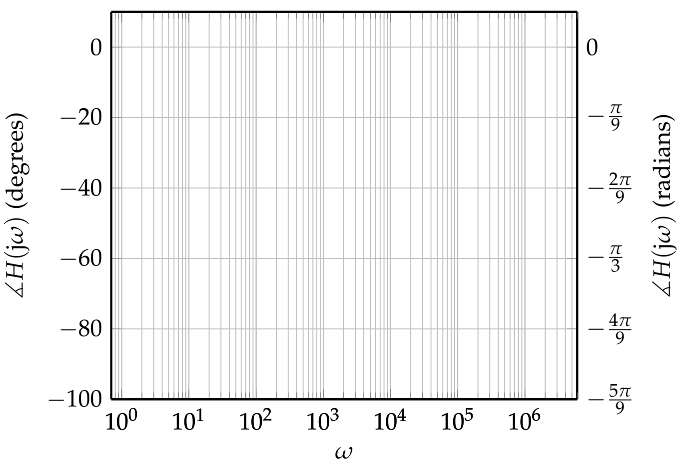
NOTE: This scenario is common in applications; usually, the data collected is in "regular" frequencies, but the analysis requires angular frequencies.

(c) To achieve your goal of keeping the voice tone but rejecting the noise from the power-lines and fluorescent lights, **at what frequency do you want to have the cutoff frequency for the lowpass filters**?

(HINT: To arrive at a unique solution consider computing the geometric mean (the analogous quantity to the arithmetic mean on a log scale) of the two frequencies of interest.)

(d) **Draw the Bode plot (straight-line approximations to the transfer function) for the magnitude (using $20 \log |H(\mathrm{j}\omega)|$) and phase of the lowpass filter.**





(e) To achieve your goal of keeping the voice tone but rejecting the noise from the power-lines and fluorescent lights, **at what frequency do you want to have the cutoff frequency for the highpass filters**?

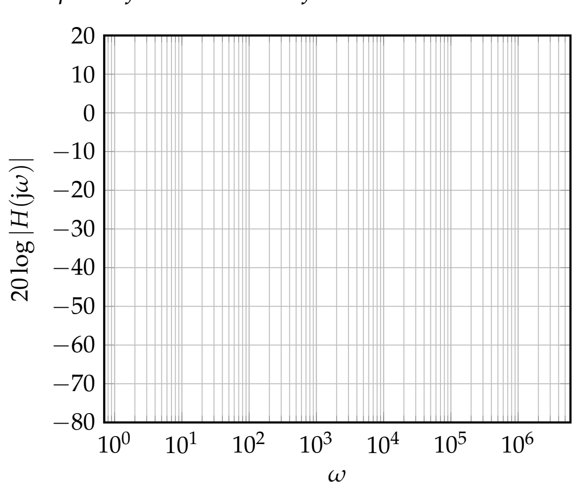
(HINT: To arrive at a unique solution consider computing the geometric mean (the analogous quantity to the arithmetic mean on a log scale) of the two frequencies of interest.)

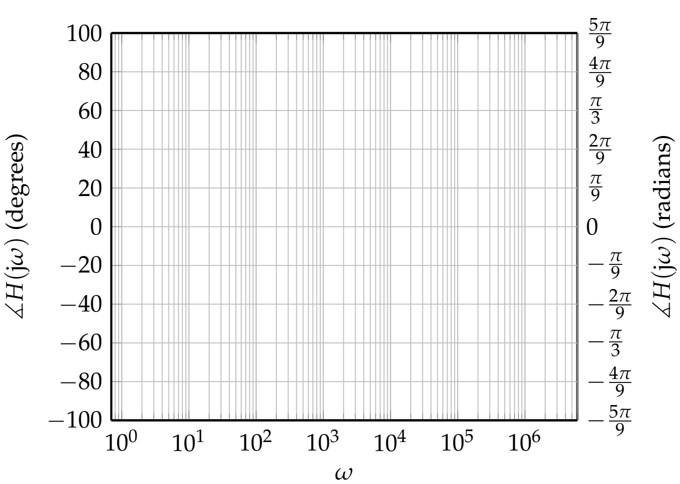
(f) **Draw the Bode plot (straight-line approximations to the transfer function) for the magnitude (using $20 \log |H(j\omega)|$) and phase of the highpass filter.**

(g) For the following questions, assume your cut-off frequencies for lowpass and highpass are 6 kHz and 189 Hz respectively. Suppose that you only had 1 µF capacitors to use. **What resistance values would you choose for your highpass and lowpass filters so that they have the desired cutoff frequencies?**

(h) The overall bandpass filter that is created by cascading the lowpass and highpass with ideal buffers in between. **Draw the Bode plot (straight-line approximations to the transfer function) for the magnitude and phase of the overall bandpass transfer function.**

(HINT: You should think about how the Bode plot of a cascade of two filters can be derived based on the Bode plots of the lower-level filters.)





(i) Suppose that the bandpass filter does not have enough suppression at 60 Hz and 60 kHz. You decide to use a cascade of three bandpass filters (with unity-gain buffers in between) (as shown in Figures 1 and 2). **What are the phasors for each of the frequency tones after all three bandpass filters?**

(HINT: Remember how you determined the transfer function of the bandpass filter from the transfer functions of the lowpass and highpass filters.)

Feel free to use a computer to help you evaluate both the magnitudes and the phases here.

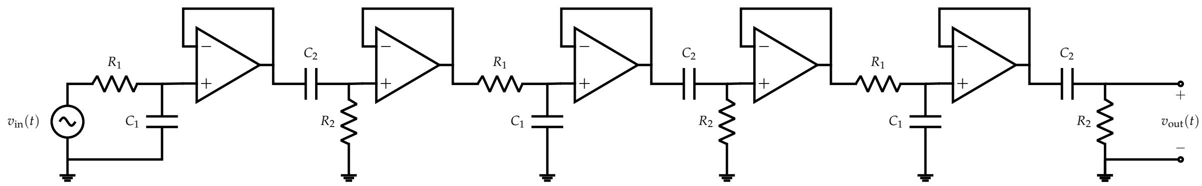


Figure 1: "Time-domain" circuit: Cascade of the three bandpass filters, using buffers to avoid loading.

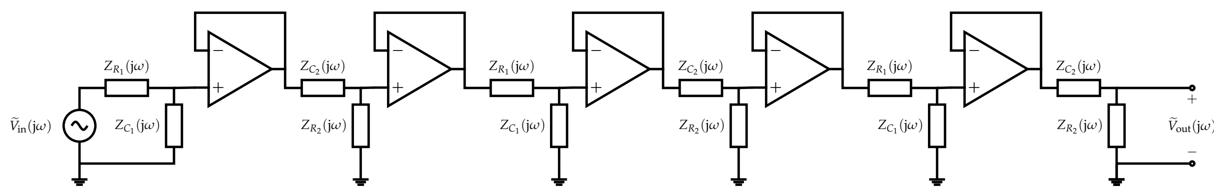
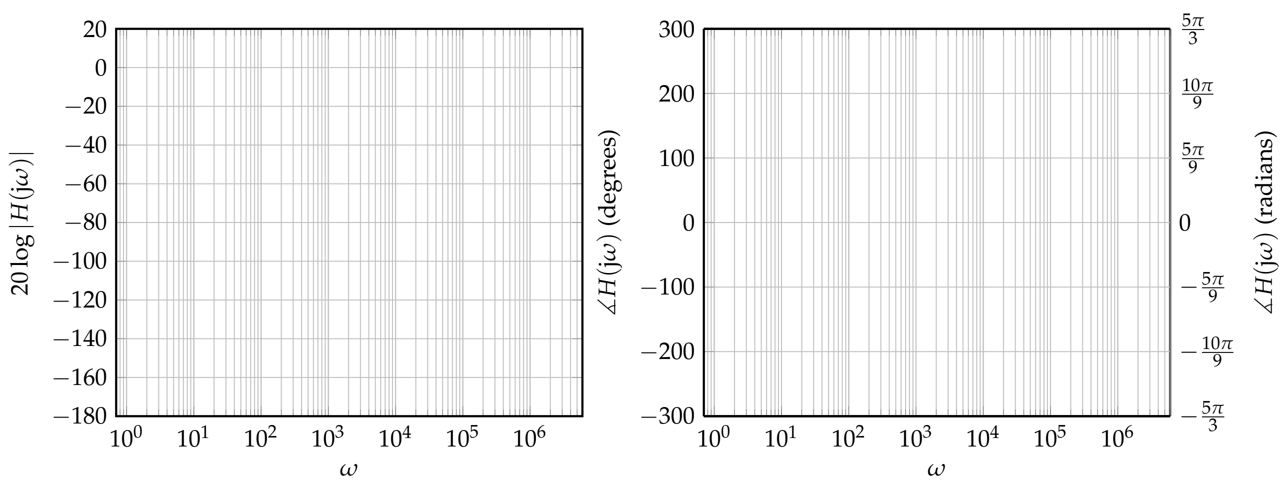


Figure 2: "Phasor-domain" circuit: Cascade of the three bandpass filters, using buffers to avoid loading.

(j) **Draw the Bode plots (straight-line approximations to the transfer function) for the magnitude and phase of the** 3^{rd} **order bandpass filter.** To highlight the difference between the 3^{rd} and 1^{st} order filters, please draw both Bode plots on a single figure.



(k) **Write the final time domain voltage waveform that would be present after the filter.**

(l) The included Jupyter notebook `filter_cascade.ipynb` sets up the same problem described above. In the notebook, you can use the slider bars to play around with:

- highpass cutoff frequency $f_{highpass}$ (i.e., the knee frequency of the highpass filters)

- lowpass cutoff frequency f_{lowpass} (i.e., the knee frequency of the lowpass filters)
- Filter order N. Filter order means the number of lowpass filters and highpass filters that are used in a row. Here, N means that there are N lowpass filters and N highpass filters, so the overall order of the entire filter is actually $2N$.

The notebook will plot the magnitude and phase, the input voltage waveform, and the output waveform at the end of the filter.

Play around with the values for the highpass and lowpass cutoff frequencies, and N.

Observe the waveforms at the output of the filter. **Comment on the values of f_{lowpass}, f_{highpass}, and N that you can use to successfully isolate the desired $600\,\text{Hz}$ tone. What happens if you keep f_{lowpass} and f_{highpass} constant, and just increase N?**

3. Circuit Design

Unlocked by Lectures 8 and 9

In this problem, you will find a circuit where several components have been left *blank* for you to fill in.

Assume that the op-amp is *ideal*. A special note on op amps in frequency domain analysis: The op-amps you learned about in 16A can be used in exactly the same way for setting up differential equations and even Phasor analysis in 16B. Treat them as ideal op-amps and invoke the Golden Rules.

You have at your disposal *only one of each* of the following components (not including R_1):

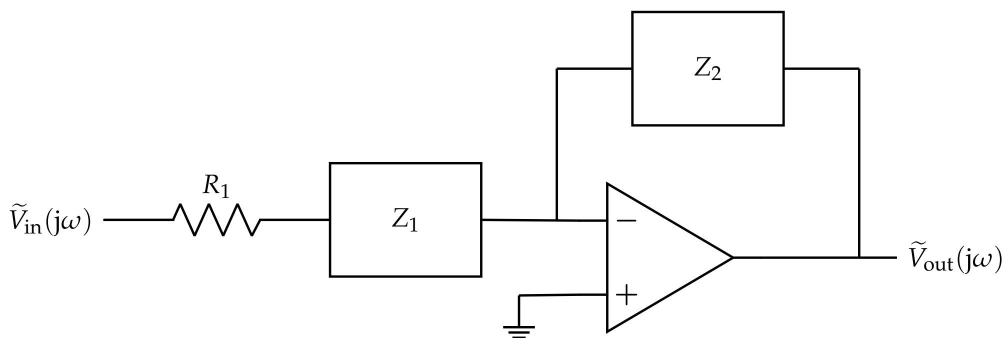
(a) an open circuit

(b) a short circuit

(c) a resistor (you choose from the values $R = 1\,\text{k}\Omega, 10\,\text{k}\Omega, 20\,\text{k}\Omega$)

(d) a capacitor (you choose from the values $C = 0.5\,\mu\text{F}, 1\,\mu\text{F}, 2\,\mu\text{F}$)

Consider the circuit below. The labeled voltages $\widetilde{V}_{\text{in}}(\text{j}\omega)$ and $\widetilde{V}_{\text{out}}(\text{j}\omega)$ are the phasor representations of $v_{\text{in}}(t)$ and $v_{\text{out}}(t)$ respectively, where $v_{\text{in}}(t)$ has the form $v_{\text{in}}(t) = v_0 \cos(\omega t + \phi)$. The transfer function $H(\text{j}\omega)$ is defined as $H(\text{j}\omega) = \frac{\widetilde{V}_{\text{out}}(\text{j}\omega)}{\widetilde{V}_{\text{in}}(\text{j}\omega)}$.



(a) Let $Z_1(\text{j}\omega)$ and $Z_2(\text{j}\omega)$ are the impedances of the boxes shown in the circuit diagram. **Write the expression of the transfer function $H(\text{j}\omega)$.**

(b) Let R_1 be $1\,\text{k}\Omega$. We have to find Z_1 and Z_2, such that the circuit's transfer function $H(\text{j}\omega)$ has the following properties:

- It is a high-pass filter.
- $|H(\text{j}\infty)| = 10$.
- $|H(\text{j}10^3)| = \sqrt{50}$.

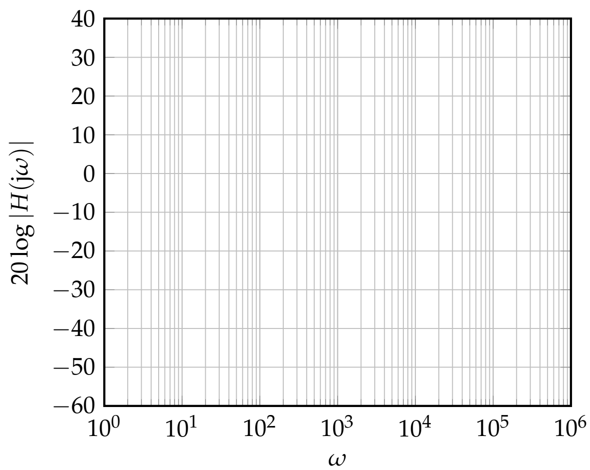
Using the fact that the circuit is a high pass filter, infer the components (we will find values later) of Z_1 and Z_2. Write the transfer function $H(\text{j}\omega)$ using these components.

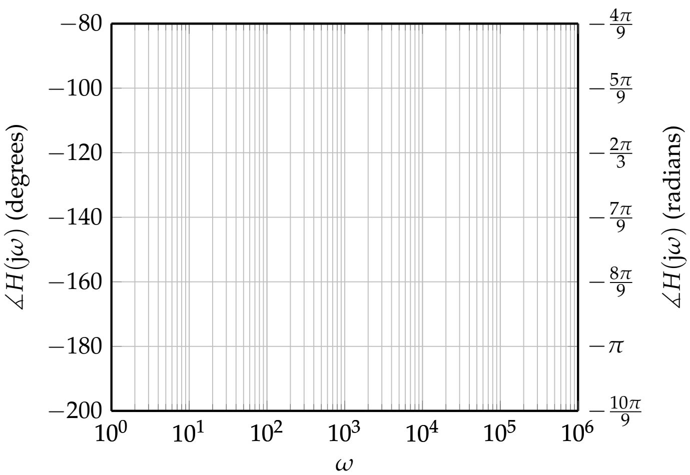
Hint: Try method of elimination: figure out what Z_2 cannot be. Once you find what Z_2 is, what does Z_1 have to be for the circuit to be a filter?

(c) **Now use the facts that $|H(\text{j}\infty)| = 10$ and $R_1 = 1\,\text{k}\Omega$ to find the component value of Z_2.**

(d) **Finally use the fact that $|H(\mathrm{j}10^3)| = \sqrt{50}$ and the values of R_1 and Z_2 to find the component value of Z_1.**

(e) **Draw the magnitude and phase Bode plots (straight-line approximations to the transfer function) of this transfer function.** Blank plots are provided here for you to use.





4. System Identification

Unlocked by Lecture 12

You are given a discrete-time system as a black box. You don't know the specifics of the system but you know that it takes one scalar input and has two states that you can observe. You assume that the system is linear and of the form

$$\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{w}[i], \tag{2}$$

where $\vec{w}[i]$ is an external small unknown disturbance, $u[i]$ is a scalar input, and

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad x[i] = \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}. \tag{3}$$

You want to identify the system parameters (a_1, a_2, a_3, a_4, b_1 and b_2) from measured data. However, you can only interact with the system via a black box model, i.e., you can see the states $\vec{x}[t]$ and set the inputs $u[i]$ that allow the system to move to the next state.

(a) You observe that the system has state $\vec{x}[i] = \begin{bmatrix} x_1[i] & x_2[i] \end{bmatrix}^\top$ at time i. You pass input $u[i]$ into the black box and observe the next state of the system: $\vec{x}[i+1] = \begin{bmatrix} x_1[i+1] & x_2[i+1] \end{bmatrix}^\top$.
Write scalar equations for the new states, $x_1[i+1]$ **and** $x_2[i+1]$. Write these equations in terms of the a_i, b_i, the states $x_1[i]$, $x_2[i]$ and the input $u[i]$. Here, assume that $\vec{w}[i] = \vec{0}$ (i.e., the model is perfect).

(b) Now we want to identify the system parameters. We observe the system at the start state $\vec{x}[0] = \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix}$. We can then input $u[0]$ and observe the next state $\vec{x}[1] = \begin{bmatrix} x_1[1] \\ x_2[1] \end{bmatrix}$. We can continue this for a sequence of ℓ inputs.
Let us define an ℓ-length trajectory to be an initial condition $\vec{x}[0]$, an input sequence $u[0], \ldots, u[\ell-1]$, and the corresponding states that are produced by the system $x[1], \ldots, x[\ell]$. **Assuming that the model is perfect ($\vec{w}[i] = \vec{0}$), what is the minimum value of ℓ you need to identify the system parameters?**

(c) We now remove our assumption that $\vec{w} = 0$. We assume it is small, so the model is approximately correct and we have

$$\vec{x}[i+1] \approx A\vec{x}[i] + Bu[i]. \tag{4}$$

Say we feed in a total of 4 inputs $u[0], \ldots, u[3]$, and observe the states $\vec{x}[0], \ldots, \vec{x}[4]$. To identify the system we need to set up an approximate (because of potential, small, disturbances) matrix equation

$$DP \approx S \tag{5}$$

using the observed values above and the unknown parameters we want to find. Let our parameter vector be

$$P := \begin{bmatrix} \vec{p}_1 & \vec{p}_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \\ b_1 & b_2 \end{bmatrix} \tag{6}$$

Find the corresponding D and S to do system identification. Write both out explicitly.

(d) Now that we have set up $DP \approx S$, we can estimate a_0, a_1, a_2, a_3, b_0, and b_1. **Give an expression for the estimates of $\vec{p}_1$ and $\vec{p}_2$ (which are denoted $\hat{\vec{p}}_1$ and $\hat{\vec{p}}_2$ respectively) in terms of D and S.** Denote the columns of S as $\vec{s}_1$ and $\vec{s}_2$, so we have $S = [\vec{s}_1 \; \vec{s}_2]$. Assume that the columns of D are linearly independent. *(HINT: Don't forget that D is not a square matrix. It is taller than it is wide.) (HINT: Can we split $DP = S$ into separate equations for p_1 and p_2?)*

5. Identifying systems from their responses to known inputs

Unlocked by Lecture 12

In many problems, we have an unknown system, and would like to characterize it. One of the ways of doing so is to observe the system response with different initial conditions (or inputs). This problem is also called system identification. It is a prototypical example of a problem that today is called machine learning — inferring an underlying pattern from data, and doing so well enough to be able to exploit that pattern in some practical setting. Go through the attached Jupyter notebook `demo_system_id.ipynb` and answer the following questions.

(a) In Example 2, we assume that instead of measuring the state $\vec{x}$, we are instead measuring a transformation of the state $\vec{y} = T\vec{x}$ where T is a full rank matrix. Assume that we perform system ID on our observations $\vec{y}[i]$ to recover A_y, B_y such that $\vec{y}[i+1] = A_y\vec{y}[i] + B_y u[i]$. **How do the identified A_y and B_y matrices relate to the original A and B matrices in the dynamics of $\vec{x}$?** Remember that our original state dynamics are $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i]$.

Hint: The answer is given in the Jupyter notebook but remember to show your work.

(b) **Please share your observations on Example 2. Comment on what impact a linear transformation of the state trace has on our ability to perform system identification.**

(c) **Prove that for any full rank transformation matrix T, the eigenvalues of A_y and A from part (a) are the same.**

(d) **Please share your observations on Example 3. Comment on the impact that changing the noise magnitude, number of samples and number of states has on the system identification performance.**

(e) **Please share your observations on Example 4. Comment on the sample efficiency of this method, i.e. do you need more or less samples for accurate system identification when given scalar observations rather than the entire state vector?**

(f) **Please share your observations on Example 5. Comment on how important the model size is for this setting.**

6. Stability Criterion

Unlocked by Lectures 12 and 13

Consider the complex plane below, which is broken into non-overlapping regions A through H. The circle drawn on the figure is the unit circle $|\lambda| = 1$.

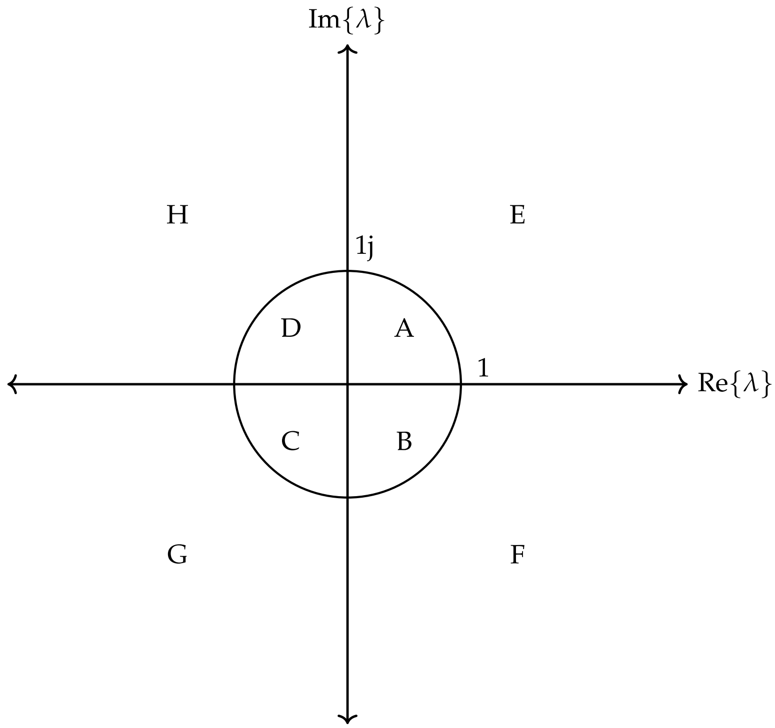


Figure 4: Complex plane divided into regions.

Consider the continuous-time system $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + v(t)$
and the discrete-time system $y[i+1] = \lambda y[i] + w[i]$. Here $v(t)$ and $w[i]$ are both disturbances to their respective systems.

In which regions can the eigenvalue λ be for the system to be *stable*? Fill out the table below to indicate *stable* regions. Assume that the eigenvalue λ does not fall directly on the boundary between two regions.

	A	**B**	**C**	**D**	**E**	**F**	**G**	**H**
Continuous Time System $x(t)$	◯	◯	◯	◯	◯	◯	◯	◯
Discrete Time System $y[i]$	◯	◯	◯	◯	◯	◯	◯	◯

7. **Bounded-Input Bounded-Output (BIBO) Stability**

Unlocked by Lectures 12 and 13

BIBO stability is a system property where bounded inputs lead to bounded outputs. It's important because we want to certify that, provided our system inputs are bounded, the outputs will not "blow up". In this problem, we gain a better understanding of BIBO stability by considering some simple continuous and discrete systems, and showing whether they are BIBO stable or not.

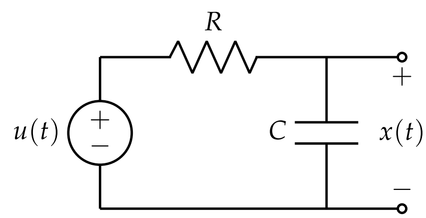
Recall that for the following simple scalar differential equation, we have the corresponding solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = ax(t) + bu(t) \qquad x(t) = \mathrm{e}^{at}x(0) + \int_0^t \mathrm{e}^{a(t-\tau)}bu(\tau)\,\mathrm{d}\tau\,. \tag{7}$$

And for the following discrete system, we have the corresponding solution:

$$x[i+1] = ax[i] + bu[i] \qquad x[i] = a^i x[0] + \sum_{k=0}^{i-1} a^k bu[i-1-k] \tag{8}$$

(a) Consider the circuit below with $R = 1\Omega$, $C = 0.5\mathrm{F}$. Let $x(t)$ be the voltage over the capacitor.



This circuit can be modeled by the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -2x(t) + 2u(t) \tag{9}$$

Intuitively, we know that the voltage on the capacitor can never exceed the (bounded) voltage from the voltage source, so this system is BIBO stable. **Show that this system is BIBO stable, meaning that $x(t)$ remains bounded for all time if the input $u(t)$ is bounded. Equivalently, show that if we assume $|u(t)| < \epsilon,\ \forall t \geq 0$ and $|x(0)| < \epsilon$, then $|x(t)| < M,\ \forall t \geq 0$ for some positive constant M.** Thinking about this helps you understand what bounded-input-bounded-output stability means in a physical circuit.

(HINT: eq. (7) may be useful. You may want to write the expression for $x(t)$ in terms of $u(t)$ and $x(0)$ and then take the norms of both sides to show a bound on $|x(t)|$. Remember that norm in 1D is absolute value. Some helpful formulas are $|ab| = |a||b|$, the triangle inequality $|a+b| \leq |a| + |b|$, and the integral version of the triangle inequality $\left|\int_a^b f(\tau)\,\mathrm{d}\tau\right| \leq \int_a^b |f(\tau)|\,\mathrm{d}\tau$, which just extends the standard triangle inequality to an infinite sum of terms.)

(b) Assume $x(0) = 0$. **Show that the system eq. (7) is BIBO unstable when $a = \mathrm{j}2\pi$ by constructing a bounded input that leads to an unbounded $x(t)$.**

It can be shown that the system eq. (7) is unstable for any purely imaginary a by a similar construction of a bounded input.

(c) Consider the discrete-time system and its solution in eq. (8). **Show that if $|a| > 1$, then even if $x[0] = 0$, a bounded input can result in an unbounded output, i.e. the system is BIBO unstable.** *(HINT: The formula for the sum of a geometric sequence may be helpful.)*

(d) Consider the discrete-time system

$$x[i+1] = -3x[i] + u[i]. \tag{10}$$

Is this system stable or unstable? Give an initial condition $x(0)$ and a sequence of non-zero inputs for which the state $x[i]$ will always stay bounded. *(HINT: See if you can find any input pattern that results in an oscillatory behavior.))*

8. **Eigenvalue Placement through State Feedback**

Unlocked by Lectures 12, 13, and 14

Consider the following discrete-time linear system:

$$\vec{x}[i+1] = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]. \tag{11}$$

In standard language, we have $A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the form: $\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]$.

(a) **Is this system controllable?**

(b) **Is this discrete-time linear system stable in open loop (without feedback control)?**

(c) Suppose we use state feedback of the form $u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] = F\vec{x}[i]$.
Find the appropriate state feedback constants, f_1, f_2 so that the state space representation of the resulting closed-loop system has eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.

(d) We are now ready to go through some numerical examples to see how state feedback works. Consider the first discrete-time linear system. Enter the matrix A and vector $\vec{b}$ from (a) for the system $\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i] + \vec{w}[i]$ into the Jupyter notebook "`eigenvalue_placement.ipynb`" and use the randomly generated $\vec{w}[i]$ as the disturbance introduced into the state equation. Observe how the norm of $\vec{x}[i]$ evolves over time for the given A. **What do you see happening to the norm of the state?**

(e) Add the feedback computed in part (c) to the system in the notebook and **explain how the norm of the state changes.**

(f) Now suppose we've got a different system described by the controlled scalar difference equation $z[i+1] = z[i] + 2z[i-1] + u[i]$. To convert this second-order discrete time system to a two-dimensional first-order discrete time system, we will let $\vec{y}[i] = \begin{bmatrix} z[i-1] \\ z[i] \end{bmatrix}$.
Write down the system representation for $\vec{y}$ in the following matrix form:

$$\vec{y}[i+1] = A_y\vec{y}[i] + \vec{b_y}u[i]. \tag{12}$$

Specify the values of the matrix A_y and the vector $\vec{b_y}$.

(g) It turns out that the original $\vec{x}[i]$ system can be converted to the $\vec{y}[i]$ system using a change of basis P. Let this coordinate change be written as $\vec{y}[i] = P\vec{x}[i]$. **First express A_y and $\vec{b_y}$ symbolically in terms of A, $\vec{b}$, and P. Then, confirm numerically that $P = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ is the correct change of basis matrix between the two systems.**

(h) For the $\vec{y}$ system from part (f), **design a feedback gain matrix $\begin{bmatrix} \overline{f}_1 & \overline{f}_2 \end{bmatrix}$ to place the closed-loop eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. Additionally, confirm that this matrix is just a change of basis of the gain matrix from part (c), i.e. $\begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} \overline{f}_1 & \overline{f}_2 \end{bmatrix} P$.**
Note that this means you can solve for the closed-loop gains of your system in any basis, and then transform it to the basis you care about.

9. Homework Process, Study Group, and Course Weekly Survey

Citing sources and collaborators are an important part of life, including being a student!

We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

At the same time, we want to check-in weekly regarding Discussions, Lectures, Lab, and Office Hours and see how effective they have all been for you as students.

Please fill out this survey link. For your submission, please attach a screenshot indicating that you have completed the survey this week.

Contributors:

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- Sally Hui.
- Alex Devonport.
- Tanmay Gautam.
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