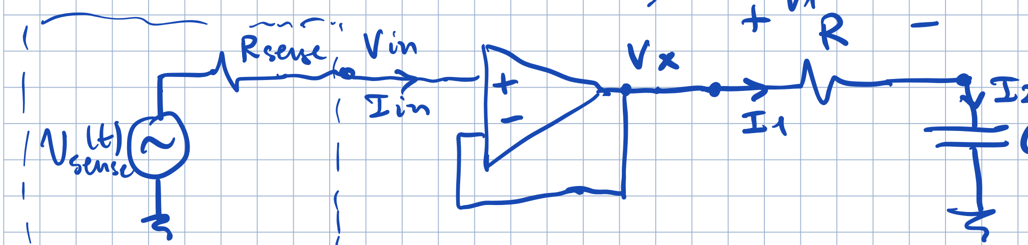
Lecture 4

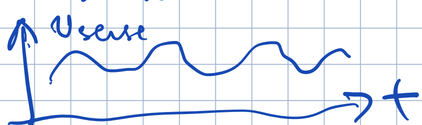
* Sensing
  * Non-homogeneous diff. eqns. with continuous inputs

Signal source (neural signal / electrode) voice / mic-board

Want this circuit to filter / separate low-frequency signal of interest from high-freq interference



Thevenin model of the sensor signal source



$V_{in} = V_{sense}$ (b/c $I_{in} = 0$)

$V_x = V_{in}$ (buffer)

$V_1 = V_x - V_{out}$

KCL: $I_1 = I_2$

Elements: $V_1 = I_1 \cdot R$

$$I_2 = C \frac{dV_{out}}{dt}$$

From KCL & Elements:

$$\frac{V_1}{R} = C \frac{dV_{out}}{dt}$$

$$\frac{V_x - V_{out}}{R} = C \frac{dV_{out}}{dt}$$

(also directly from NVA nodal voltage analysis)

$$\frac{V_{sense} - V_{out}}{R} = C \frac{dV_{out}}{dt}$$

(b.c. $V_x = V_{in} = V_{sense}$)

$$\boxed{\frac{dV_{out}(t)}{dt} = -\frac{V_{out}(t)}{RC} + \frac{V_{sense}(t)}{RC}}$$

$V_{sense}(t)$ is a continuous time signal

In disc. & hw:

$$(1) \quad V_{out}(t) = \underbrace{V_{out}(0) \cdot e^{-\frac{t}{RC}}}_{\text{homogeneous solution}} + \underbrace{\frac{1}{RC} \int_0^t V_{sense}(\theta)\, e^{-\frac{1}{RC}(t-\theta)}\, d\theta}_{\text{non-homogeneous soln.}}, \quad t \geq 0$$

"response to initial condition"

"response to continuous time input"

The circuit "computes" this solution ?

[General form: $\frac{d}{dt}x(t) = \lambda x(t) - \lambda u(t)$]

For our circuit $\lambda = -\frac{1}{RC}$, $u(t) = V_{sense}$, $x(t) = V_{out}$

Let's try to compute the response for $u(t)$'s of interest.

<u>Example 1</u>: $u(t) = e^{st}$, $s \neq 0$

$$(2) \quad \boxed{\frac{d}{dt}x(t) = \lambda x(t) - \lambda u(t)}, \quad t \geq 0$$

Can solve:

$$x(t) = x(0)\, e^{\lambda t} - \lambda \int_0^t u(\theta)\, e^{\lambda(t-\theta)}\, d\theta$$

$$= x(0)\, e^{\lambda t} - \lambda e^{\lambda t} \int_0^t \underbrace{e^{s\theta} \cdot e^{-\lambda\theta}}_{e^{(s-\lambda)\theta}}\, d\theta$$

$\vdots$

or Guess & check:

$$x(t) = K e^{st} \quad , \quad t \geq 0$$

From (2)

$$K \cdot s \cdot \cancel{e^{st}} = \lambda \cdot K \cdot \cancel{e^{st}} - \lambda \cancel{e^{st}}$$

$$K s = \lambda K - \lambda$$

$$K(s-\lambda) = -\lambda$$

$$K = -\frac{\lambda}{s-\lambda} \Rightarrow x_\lambda(t) = -\frac{\lambda}{s-\lambda} e^{st}$$

To complete, add a homogeneous soln.

$$x(t) = K_2 e^{\lambda t} + K \cdot e^{st}$$

$$x(0) = K_2 + K \Rightarrow K_2 = x(0) - K$$

$$= x(0) + \frac{\lambda}{s-\lambda}$$

$$\boxed{x(t) = \left(x(0) + \frac{\lambda}{s-\lambda}\right) e^{\lambda t} - \frac{\lambda}{s-\lambda} e^{st}}$$

Example 2: $u(t) = \cos(\omega t)$

$$\frac{d}{dt} x(t) = \lambda x(t) - \lambda u(t)$$

$$\boxed{x(t) = x(0) e^{\lambda t} - \lambda \int_0^t u(\theta) e^{\lambda(t-\theta)} d\theta \quad , t \geq 0}$$

$$= x(0) e^{\lambda t} - \lambda \int_0^t \cos(\omega \theta) e^{\lambda(t-\theta)} d\theta$$

$$= x(0) e^{\lambda t} - \lambda e^{\lambda t} \int_0^t \cos(\omega\theta) e^{-\lambda\theta} d\theta$$

[from calc:

$$\int \cos(bx) e^{ax} dx = \frac{e^{ax}}{a^2+b^2} (b \sin(bx) + a \cos(bx))$$

$$x(t) = x(0) e^{\lambda t} - \lambda e^{\lambda t} \Big(\frac{e^{-\lambda t}}{\lambda^2+\omega^2} (\omega \sin(\omega t) - \lambda \cos(\omega t)) - \frac{1}{\lambda^2+\omega^2} (0 - \lambda) \Big)$$

$$x(t) = x(0) e^{\lambda t} - \frac{\lambda}{\lambda^2+\omega^2} (\omega \sin(\omega t) - \lambda \cos(\omega t)) - \frac{\lambda^2}{\lambda^2+\omega^2} e^{\lambda t}$$

$$x(t) = \underbrace{\left(x(0) - \frac{\lambda^2}{\lambda^2+\omega^2} \right) e^{\lambda t}}_{1} \underbrace{- \frac{\lambda}{\lambda^2+\omega^2} (\omega \sin(\omega t) - \lambda \cos(\omega t))}_{2}$$

$t \to \infty$ (steady-state), $\lambda < 0$

(1) $\to 0$ b.c. $\lambda < 0$ (in RC $\lambda = -\frac{1}{RC}$)

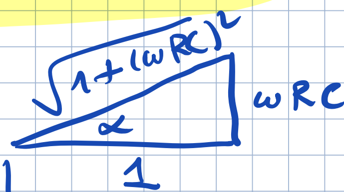
(2) for $\lambda = -\frac{1}{RC}$

$$x(t) = \frac{\frac{1}{RC}\,\omega \sin(\omega t) + \left(\frac{1}{RC}\right)^2 \cos(\omega t)}{\left(\frac{1}{RC}\right)^2 + \omega^2}$$

(2)

$$x(t) = \frac{\omega RC \sin(\omega t) + \cos(\omega t)}{1 + (\omega RC)^2}$$

Remember the trig identity:



$$\boxed{\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha-\beta)}$$

$\alpha = \mathrm{atan2}(\omega RC, 1)$

$$x(t) = \frac{1}{\sqrt{1+(\omega RC)^2}}\left(\underbrace{\frac{1}{\sqrt{1+(\omega RC)^2}}}_{\cos(\alpha)}\cdot\cos(\omega t) + \underbrace{\frac{\omega RC}{\sqrt{1+(\omega RC)^2}}}_{\sin(\alpha)}\cdot\sin(\omega t)\right)$$

$$x(t) = \frac{1}{\sqrt{1+(\omega RC)^2}}\cos(\omega t - \alpha)$$

$\theta = -\alpha$

$$x(t) = \frac{1}{\sqrt{1+(\omega RC)^2}}\cos(\omega t + \theta)$$

, $= -\mathrm{atan2}(\omega RC, 1)$

Remember: $u(t) = \cos(\omega t)$, ω – angular frequency [rad/s]

$\omega = 2\pi \cdot f = 2\pi \cdot \frac{1}{T}$, where T is the period.

Case 1: $\omega \gg \frac{1}{RC} \Rightarrow \omega RC \gg 1$

$$x(t) \approx 0$$

Case 2: $\omega \ll \frac{1}{RC} \Rightarrow \omega RC \ll 1$

$$x(t) = \cos(\omega t + \theta)$$

Our system (circuit) is a "low-pass" filter with $\frac{1}{RC}$ cut-off frequency.

Frequencies below $\frac{1}{RC}$ pass

Frequencies above $\frac{1}{RC}$ are attenuated

Can also arrive here with a guess:

Guess: $\boxed{x(t) = A\cos(\omega t + \theta)}$ do this for: $u(t) = V_{sense}\cos(\omega t)$

System: $\frac{d}{dt}x(t) = \lambda x(t) - \lambda u(t)$

$$-A\omega \sin(\omega t + \theta) = \lambda A \cos(\omega t + \theta) - \lambda V_{sense}\cos(\omega t)$$

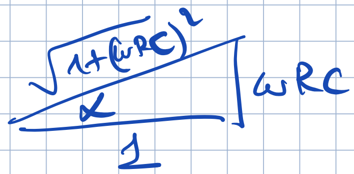
$$V_{sense}\cos(\omega t) = A\cos(\omega t + \theta) + \frac{A\omega}{\lambda}\sin(\omega t + \theta)$$

$$= A\left(\cos(\omega t + \theta) + \frac{\omega}{\lambda}\sin(\omega t + \theta)\right)$$

$$\lambda = -\frac{1}{RC}$$

$$V_{sense}\cos(\omega t) = A\left(1\cdot\cos(\omega t+\theta) - \omega RC\sin(\omega t+\theta)\right)$$

$$V_{sense}\cdot\cos(\omega t) = A\sqrt{1+(\omega RC)^2}\left(\underbrace{\frac{1}{\sqrt{1+(\omega RC)^2}}}_{\cos(\alpha)}\cos(\omega t+\theta) - \underbrace{\frac{\omega RC}{\sqrt{1+(\omega RC)^2}}}_{\sin(\alpha)}\sin(\omega t+\theta)\right)$$



$$\alpha = \mathrm{atan2}(\omega RC, 1)$$

$$V_{sense}\cos(\omega t) = A\sqrt{1+(\omega RC)^2}\left(\cos(\alpha)\cos(\omega t+\theta) - \right.$$
$$\left. - \sin(\alpha)\sin(\omega t+\theta)\right)$$

$$\boxed{V_{sense}\cos(\omega t) = A\sqrt{1+(\omega RC)^2}\cos(\omega t+\theta+\alpha)}$$

$$V_{sense} = A\cdot\sqrt{1+(\omega RC)^2} \quad \& \quad \cancel{\omega t} = \omega t+\theta+\alpha$$

$$\Downarrow \qquad\qquad \theta+\alpha = 0$$

$$\boxed{A = \frac{V_{sense}}{\sqrt{1+(\omega RC)^2}}} \qquad \boxed{\theta = -\alpha = -\mathrm{atan2}(\omega RC, 1)}$$

Back to our guess: $x(t) = A\cos(\omega t+\theta)$

solution:

$$\boxed{x_d(t) = \frac{V_{sense}}{\sqrt{1+(\omega RC)^2}}\cos(\omega t+\theta) \qquad \theta = -\mathrm{atan2}(\omega RC, 1)}$$

more general:

$$x(t) = x(t_0)\,e^{\lambda(t-t_0)} - \lambda\int_{t_0}^{t} u(\theta)\,e^{\lambda(t-\theta)}\,d\theta \quad , t \geq t_0$$