Lecture 6

* Systems of differential equations
* Diagonalization

Have a system:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t), \quad \vec{x}(0), \text{ want } \vec{x}(t) \text{ for } t \geq 0$$

Original coordinates:

Nice coordinates:

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{x} \arrow[d, bend right, "V^{-1}"'] \\
\tilde{\vec{x}} \arrow[r, "\underline{V^{-1}AV}"', "?"'] \arrow[u, bend right, "V"'] & \frac{d}{dt}\tilde{\vec{x}} \arrow[u, bend right, "V"']
\end{tikzcd}$$

(a) $\vec{x} = V\tilde{\vec{x}} = [\vec{v}_1 \ldots \vec{v}_n]\begin{bmatrix}\tilde{x}_1\\ \vdots \\ \tilde{x}_n\end{bmatrix} = \tilde{x}_1\vec{v}_1 + \ldots + \tilde{x}_n\vec{v}_n$

$\tilde{x}_1, \ldots, \tilde{x}_n$ coordinates of $\vec{x}$ in basis V.

(b) $\tilde{\vec{x}} = V^{-1}\vec{x}$

$$\frac{d}{dt}\tilde{\vec{x}} \overset{(b)}{=} \frac{d}{dt}(V^{-1}\vec{x}) = V^{-1}\frac{d}{dt}\vec{x} = V^{-1}(A\vec{x} + B\vec{u}) \overset{(a)}{=}$$

$$= V^{-1}(AV\tilde{\vec{x}} + B\vec{u}) = V^{-1}AV\tilde{\vec{x}} + V^{-1}B\vec{u}$$

step 1

$$\boxed{\frac{d}{dt}\tilde{\vec{x}} = V^{-1}AV\tilde{\vec{x}} + V^{-1}B\vec{u}}$$

step 2 Get $\tilde{\vec{x}}(0)$ from (b) $\boxed{\tilde{\vec{x}}(0) = V^{-1}\vec{x}(0)}$

& solve for $\tilde{\vec{x}}(t)$ when $V^{-1}AV$ to be <u>upper-triangular</u> or <u>diagonal</u>.

step 3 Go back to $\vec{x}(t)$ using (a)

$$\boxed{\vec{x}(t) = V\,\vec{\tilde{x}}(t)}$$

Want: $V^{-1}AV = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ (Diagonal matrix) $\Downarrow$ will give us separable scalar diff. eqns. in $\vec{\tilde{x}}$.

<u>How to figure out which V to use?</u>

$V = [\vec{v_1} \; \dots \; \vec{v_n}]$

$$V^{-1}AV = V^{-1}\left(A[\vec{v_1} \dots \vec{v_n}]\right)$$

$$= V^{-1}[A\vec{v_1} \; \dots \; A\vec{v_n}]$$

$$\stackrel{*}{=} V^{-1}[\lambda_1\vec{v_1} \; \dots \; \lambda_n\vec{v_n}]$$

[* From 16 A: $A\vec{v_i} = \lambda_i\vec{v_i}$ — eigenvector, eigenvalue]

if $\vec{v_i}$'s are lin. indep eigenvectors of A.

$$[\lambda_1\vec{v_1} \dots \lambda_n\vec{v_n}] = \underbrace{[\vec{v_1} \dots \vec{v_n}]}_{V} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\Lambda} = V \cdot \Lambda$$

$$V^{-1}AV = \underbrace{V^{-1}V}_{I}\Lambda = \Lambda \text{ - diagonal}$$ ☺

so if V is an eigenbasis (basis of eigenvectors) then:

$V^{-1}AV$ - diagonal

$$\frac{d}{dt}\vec{\tilde{x}}(t) = \Lambda\vec{\tilde{x}}(t) + V^{-1}B\vec{u}(t)$$

a collection of separable scalar diff. eqns.

Want to find λ_i's (eigenvalues) & $\vec{v}_i$'s (eigenvectors) of A to compose V and Λ.

Example: Our 2^{nd}-order RC circuit

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

[Recall: $A\vec{v} = \lambda\vec{v}$

$\underbrace{(A-\lambda I)}_{\text{has a nullspace}}\vec{v} = \vec{0}$

$\det(A-\lambda I) = 0$ or $\det(\lambda I - A) = 0$ if λ is an eigenvalue]

① Find eigenvalues of A:

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda+5 & -2 \\ -2 & \lambda+2 \end{bmatrix}\right)$$

$$= (\lambda+5)(\lambda+2) - 4 = \lambda^2 + 7\lambda + 6 = (\lambda+6)(\lambda+1) = 0$$

$$\underbrace{\lambda_1 = -1, \quad \lambda_2 = -6}$$

$$\boxed{\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}}$$

② Find eigenvectors:

$A - \lambda I$ Null-space for $A - \lambda_1 I$? $\lambda_1 = -1$

$$A - \lambda_1 I = \begin{bmatrix} -5+1 & 2 \\ 2 & -2+1 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}}_{A-\lambda_1 I} \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{v}_1} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{0}} \Rightarrow \boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$\lambda_2 = -6$ Null-space for $A - \lambda_2 I = ?$

$$A - \lambda_2 I = \begin{bmatrix} -5+6 & 2 \\ 2 & -2+6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}_{A - \lambda_2 I} \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\vec{v_2}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{0}} \Rightarrow \boxed{\vec{v_2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

$$\boxed{\begin{aligned} V &= \begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \\ V^{-1} &= \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} \end{aligned}}$$

3 Homogeneous solution for $\vec{\tilde{x}}(t)$:

$$\frac{d}{dt} \vec{\tilde{x}}(t) = \Lambda \vec{\tilde{x}}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\hat{x}_1(t) \\ -6\hat{x}_2(t) \end{bmatrix}$$

$$\frac{d}{dt} \begin{bmatrix} \tilde{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}\tilde{x}_1(t) \\ \frac{d}{dt}\hat{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\hat{x}_1(t) \\ -6\hat{x}_2(t) \end{bmatrix}$$

scala separable diff. eqs.
$$\begin{cases} \frac{d}{dt}\hat{x}_1(t) = -\hat{x}_1(t) & \Rightarrow \hat{x}_1(t) = \hat{x}_1(0)e^{-t} \\ \frac{dt}{dt}\hat{x}_2(t) = -6\hat{x}_2(t) & \Rightarrow \hat{x}_2(t) = \tilde{x}_2(0)e^{-6t} \end{cases}$$

$$\vec{\tilde{x}}(t) = \begin{bmatrix} \hat{x}_1(0)e^{-t} \\ \hat{x}_2(0)e^{-6t} \end{bmatrix}$$

Important: $\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0)$

$$\vec{\tilde{x}}(0) = \underbrace{\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}}_{V^{-1}} \cdot \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{x}(0)} = \begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \end{bmatrix}$$

$$\boxed{\vec{\tilde{x}}(t) = \begin{bmatrix} \frac{3}{5}e^{-t} \\ -\frac{1}{5}e^{-6t} \end{bmatrix}}$$

but want $\vec{x}(t)$

4 $\vec{x}(t) = V\vec{\tilde{x}}(t) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{5}e^{-t} \\ -\frac{1}{5}e^{-6t} \end{bmatrix}$

$$\vec{x}(t) = \begin{bmatrix} \frac{3}{5}e^{-t} + \frac{2}{5}e^{-6t} \\ \frac{6}{5}e^{-t} - \frac{1}{5}e^{-6t} \end{bmatrix}$$

(homogeneous solution $v_{in} = 0V$ for $t \geq 0$

$\Rightarrow$ no - non-homogeneous solution.

In general:
$$\frac{d}{dt}\vec{x}(t) = \Lambda\vec{\tilde{x}}(t) + \underbrace{\overbrace{V^{-1}B}^{\hat{B}}\vec{u}(t)}$$

$$\frac{d}{dt}\vec{\tilde{x}}(t) = \Lambda\vec{\tilde{x}}(t) + \vec{\tilde{u}}(t)$$

$$\vec{\tilde{x}}(t) = \vec{\tilde{x}}_h(t) + \vec{\hat{x}}_{nh}(t)$$

e.g. 2×2
$$\boxed{\vec{\tilde{x}}_{nh}(t) = \begin{bmatrix} \int_0^t \hat{u}_1(\theta)e^{\lambda_1(t-\theta)}d\theta \\ \int_0^t \hat{u}_2(\theta)e^{\lambda_2(t-\theta)}d\theta \end{bmatrix}}$$

$$\hat{B} = \begin{bmatrix} \hat{b}_{11} & \hat{b}_{12} \\ \hat{b}_{21} & \hat{b}_{22} \end{bmatrix} = V^{-1}B$$

[Remember the scalar case:
$$\frac{d}{dt}z(t) = \lambda z(t) + bu(t)$$
$$z(t) = z(0)e^{\lambda t} + b\int_0^t u(\theta)e^{\lambda(t-\theta)}d\theta$$
]

$$\vec{\tilde{u}} = V^{-1}B\vec{u} = \begin{bmatrix} \hat{b}_{11} & \hat{b}_{12} \\ \hat{b}_{21} & \hat{b}_{22} \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \hat{b}_{11}u_1(t) + \hat{b}_{12}u_2(t) \\ \hat{b}_{21}u_1(t) + \hat{b}_{22}u_2(t) \end{bmatrix} = \begin{bmatrix} \tilde{u}_1 \\ \hat{u}_2 \end{bmatrix}$$

For our RC example:

$$B = \vec{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \qquad \vec{\tilde{u}}(t) = V^{-1}\vec{b}\,V_{in}(t) = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix}V_{in}(t)$$

$$\vec{\hat{u}}(t) = \begin{bmatrix} \frac{3}{5}V_{in}(t) \\ -\frac{6}{5}V_{in}(t) \end{bmatrix} \qquad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}$$

$$\boxed{\vec{\tilde{x}}_{nh}(t) = \begin{bmatrix} \int_0^t \frac{3}{5}V_{in}(\theta)e^{-(t-\theta)}d\theta \\ \int_0^t \left(-\frac{6}{5}\right)V_{in}(\theta)e^{-6(t-\theta)}d\theta \end{bmatrix}}$$

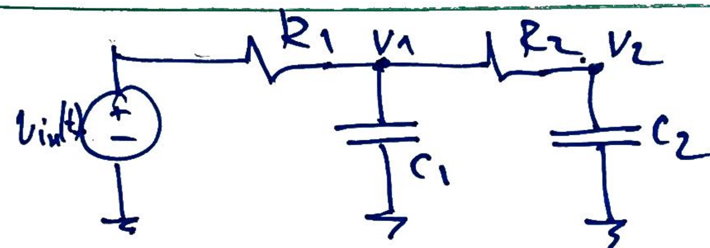
Going back:

$$\vec{x}_{nh}(t) = V \cdot \vec{\tilde{x}}_{nh}(t) \quad , \; V = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$\vec{x}_{nh}(t) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{5} e^{-t} \int_0^t v_{in}(\theta) e^{\theta} d\theta \\ -\frac{6}{5} e^{-6t} \int_0^t v_{in}(\theta) e^{6\theta} d\theta \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} \frac{3}{5} e^{-t} \int_0^t v_{in}(\theta) e^{\theta} d\theta + \frac{12}{5} e^{-6t} \int_0^t v_{in}(\theta) e^{6\theta} d\theta \\ \frac{6}{5} e^{-t} \int_0^t v_{in}(\theta) e^{\theta} d\theta - \frac{6}{5} e^{-6t} \int_0^t v_{in}(\theta) e^{6\theta} d\theta \end{bmatrix}}$$

<u>New example</u>:



for $v_{in}(t) = \begin{cases} 0V, & t<0 \\ 1V, & t \geq 0 \end{cases}$

$$\vec{x}(0) = \begin{bmatrix} v_1(0) \\ v_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{x}_h(t) = \vec{0} \quad \text{b/c. } \vec{x}(0) = 0$$

$$\vec{x}_{nh}(t) = \begin{bmatrix} \frac{3}{5} e^{-t} \int_0^t e^{\theta} d\theta + \frac{12}{5} e^{-6t} \int_0^t e^{6\theta} d\theta \\ \frac{6}{5} e^{-t} \int_0^t e^{\theta} d\theta - \frac{6}{5} e^{-6t} \int_0^t e^{6\theta} d\theta \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{5} e^{-t} (e^{t} - \cancel{e^{0}}^{\,1}) + \frac{12}{5} e^{-6t} \cdot \frac{1}{6} \cdot (e^{6t} - \cancel{e^{0}}^{\,1}) \\ \frac{6}{5} e^{-t} (e^{t} - 1) - \frac{6}{5} e^{-6t} \frac{1}{6} (e^{6t} - 1) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{5} - \frac{3}{5} e^{-t} + \frac{2}{5} - \frac{2}{5} e^{-6t} \\ \frac{6}{5} - \frac{6}{5} e^{-t} - \frac{1}{5} + \frac{1}{5} e^{-6t} \end{bmatrix}$$