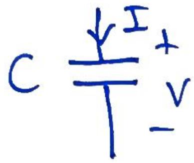
Lecture 7

* Intro to inductors
* 2^{nd} order systems with complex eigenvalues

Capacitor



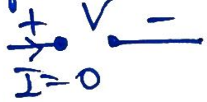
$$\boxed{I(t) = C\frac{d}{dt}V(t)}$$

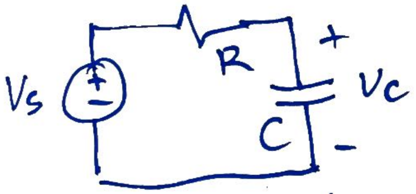
* "resists" a change in voltage
* stores energy in an electric field

$E = \frac{1}{2}CV^2$

C - unit Farad [F]

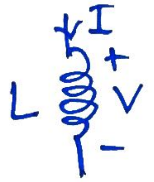
* at ~~a~~ DC it acts as an open-circuit





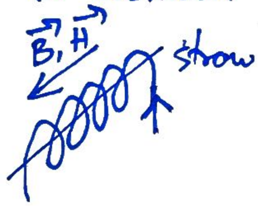
$V_C \sim e^{-\frac{t}{RC}}$, $\tau = RC$ time constant

Inductor



$$\boxed{V(t) = L\frac{d}{dt}I(t)}$$

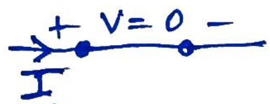
* "resists" a change in current
* stores energy in a magnetic field

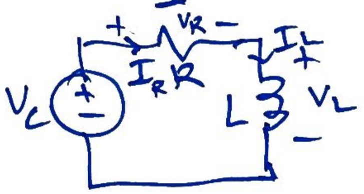


$E = \frac{1}{2}LI^2$

L - unit Henry [H]

* at constant current inductor behaves like a: short-circuit (wire)





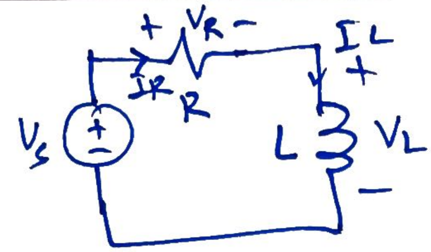
$\tau = ?$

Decaying exponential $e^{-\frac{t}{\tau}}$?

Assume:

$V_S(t<0) = 1V$

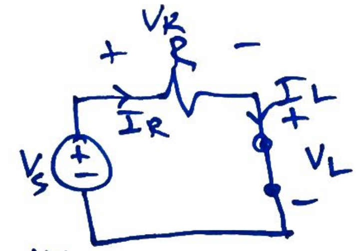
$V_S(t \geq 0) = 0V$



Solve for $I_L(t)$, $t \geq 0$

First: find $I_L(0)$

For $t<0$ => steady-state



$$I_L(t) = \frac{\overbrace{V_S(t)}^{1V} - \overbrace{V_L(t)}^{0}}{R} = \frac{1V}{R} \quad , t<0$$

since current does not change instantaneously on the inductor => $\boxed{I_L(0) = I_L(t<0) = \frac{1V}{R}}$

Next step: solve for $t \geq 0$

$$V_L(t) = L\frac{d}{dt}I_L(t) \quad , \quad I_L(t) = \frac{V_S(t) - V_L(t)}{R}$$

$$I_L(t) = \frac{\overbrace{V_S(t)}^{0}}{R} - \frac{L}{R}\frac{d}{dt}I_L(t) \quad \text{for } t \geq 0$$

$$I_L(t) = -\frac{L}{R}\frac{d}{dt}I_L(t)$$

$$\boxed{\frac{d}{dt}I_L(t) = -\frac{R}{L}I_L(t) \quad , t \geq 0}$$

$$I_L(t) = I_L(0)\,e^{-\frac{R}{L}t}$$

$$= I_L(0)\,e^{-\frac{t}{\tau}} \quad , \tau = \frac{L}{R}$$

$I_L(0) = \frac{1V}{R}$

compare that with:

$$\frac{d}{dt}V_C(t) = -\frac{1}{RC}V_C(t)$$

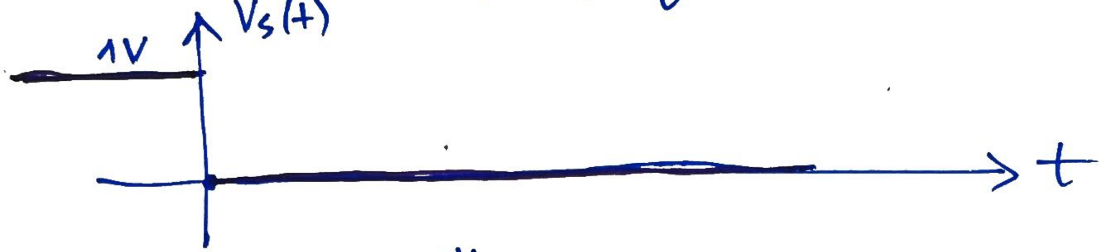
(l3)

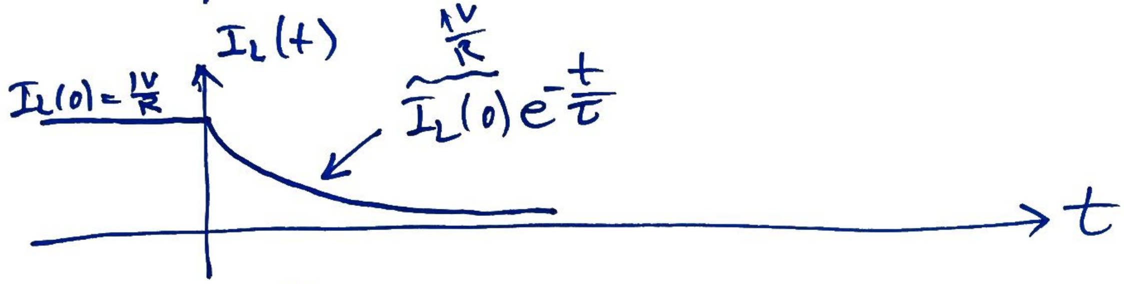
$t \geq 0$

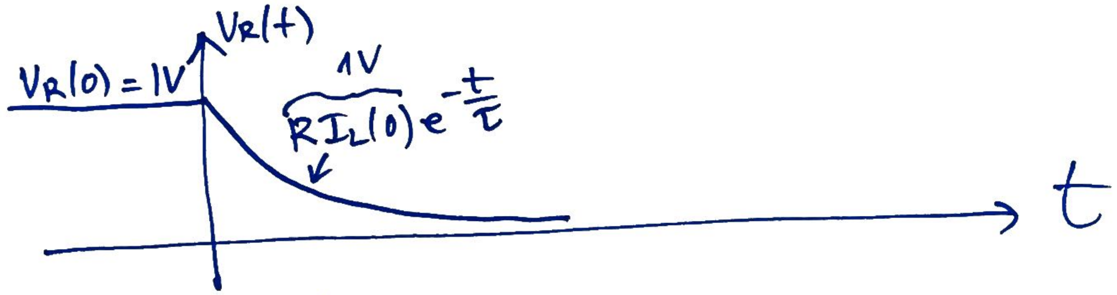
$$V_R(t) = R \cdot I_R(t) = R \cdot I_L(t) = R \cdot \underbrace{I_L(0)}_{\frac{1V}{R}} e^{-\frac{t}{\tau}}$$

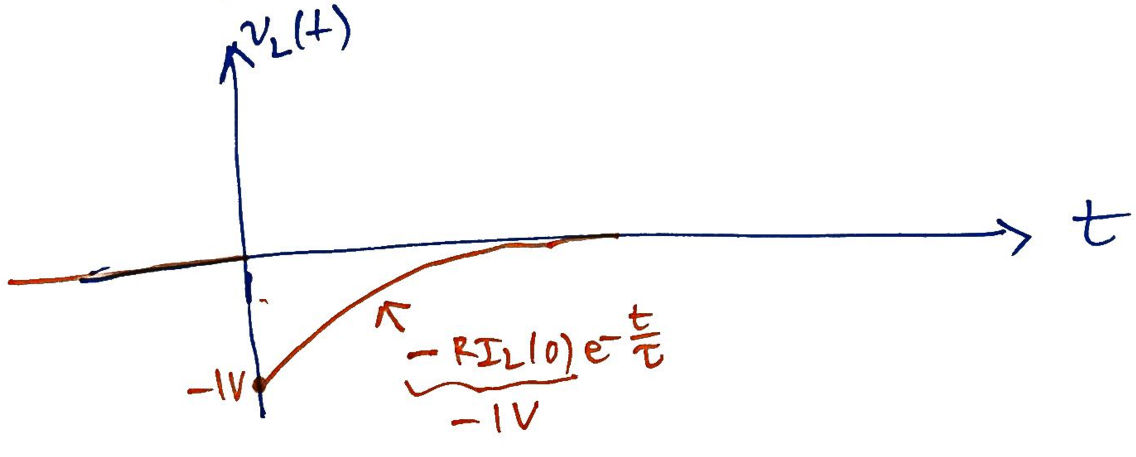
$$V_L(t) = \underbrace{V_S(t)}_{0} - V_R(t) = -V_R(t) = 1V \cdot e^{-\frac{t}{\tau}}$$

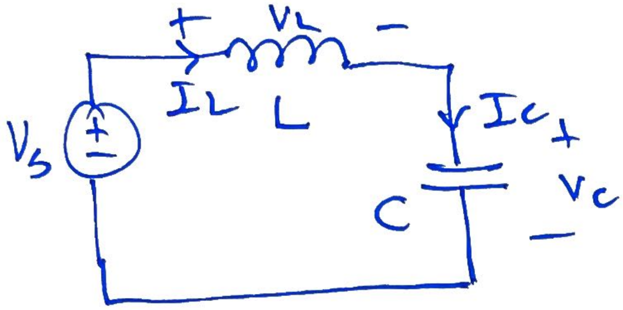
$$= -1V \cdot e^{-\frac{t}{\tau}}$$











$I_C(t) = C \frac{d}{dt} V_C(t)$

$V_L(t) = L \frac{d}{dt} I_L(t)$

$I_L(t) = I_C(t)$ (KCL)

$V_L(t) = V_S(t) - V_C(t)$

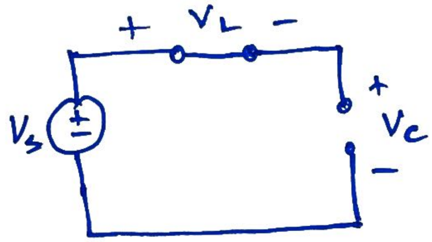
$V_S(t<0) = 1V$

$V_S(t \geq 0) = 0V$

$V_C(t) = ?$, $I_L(t) = ?$, $t \geq 0$

First: find initial conditions:

In steady-state
$t < 0$



$\Rightarrow$ $V_C(t<0) = V_S = 1V$

$I_L(t<0) = 0\,A$

$\Downarrow$

$$\boxed{\begin{aligned} V_C(0) &= 1V \\ I_L(0) &= 0\,A \end{aligned}}$$

Next: solve for $t \geq 0$

$V_S(t) = 0V \Rightarrow V_L(t) = -V_C(t)$

$I_L(t) = I_C(t)$

(1) $L \frac{d}{dt} I_L(t) = -V_C(t)$

(2) $I_L(t) = C \frac{d}{dt} V_C(t)$

$\Downarrow$

(1)
$$\boxed{\begin{aligned} \frac{d}{dt} I_L(t) &= -\frac{1}{L} V_C(t) \\ \frac{d}{dt} V_C(t) &= \frac{1}{C} I_L(t) \end{aligned}}$$

$$\underbrace{\frac{d}{dt}\begin{bmatrix} I_L(t) \\ v_c(t) \end{bmatrix}}_{\frac{d}{dt}\vec{x}(t)} = \underbrace{\begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} I_L(t) \\ v_c(t) \end{bmatrix}}_{\vec{x}(t)}$$

Compute the eigenvalues of A:

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & \frac{1}{L} \\ -\frac{1}{C} & \lambda \end{bmatrix}\right) = \lambda^2 + \frac{1}{LC} = 0$$

$$\boxed{\lambda_{1,2} = \pm j\sqrt{\frac{1}{LC}} \quad , \; j = \sqrt{-1}}$$

Assume: $L = 1H$, $C = 1F$ (artificially large values)

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = \pm j$$

want to find $\vec{v_1}$ & $\vec{v_2}$ s.t

$$A\vec{v_1} = \lambda_1 \vec{v_1} \quad \& \quad A\vec{v_2} = \lambda_2 \vec{v_2}$$

Nullspace style

$\lambda_1 = j$

$$(A - \lambda_1 I)\cdot\vec{v_1} = \vec{0}$$

$$\begin{bmatrix} -j & -1 \\ 1 & -j \end{bmatrix}\begin{bmatrix} \; \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{v_1} = \begin{bmatrix} j \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ j \end{bmatrix} \quad \begin{bmatrix} 1 \\ -j \end{bmatrix}$$

$$\boxed{\vec{v_1} = \begin{bmatrix} 1 \\ -j \end{bmatrix}}$$

$\lambda_2 = -j$

$$\vec{v_2} = \begin{bmatrix} 1 \\ ? \end{bmatrix}$$

$$A\vec{v_2} = \lambda_2 \vec{v_2}$$

Any multiple of eigenvector is also an eigenvector from eigenspace:

$$A k\vec{v_2} = \lambda_2 k \vec{v_2}$$

so can normalize one element to 1.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ x \end{bmatrix} = -j\begin{bmatrix} 1 \\ x \end{bmatrix}$$

$$-x = -j \Rightarrow x = j$$

$$\boxed{\vec{v_2} = \begin{bmatrix} 1 \\ j \end{bmatrix}}$$

Transform coordinates (represent $\vec{x}$ in V eigenbasis) L6

$$\vec{x} = V\vec{\hat{x}} = \begin{bmatrix}\vec{v_1} & \dots & \vec{v_n}\end{bmatrix}\begin{bmatrix}\hat{x}_1(t)\\ \vdots \\ \hat{x}_n(t)\end{bmatrix}$$

$$= \underset{\text{coordinate}}{\hat{x}_1(t)}\,\underset{\text{basis vector}}{\vec{v_1}} + \dots + \hat{x}_n(t)\cdot\vec{v_n}$$

Found a way to express $\vec{x}$ in V-eigenbasis with $\vec{\hat{x}}$ coordinates.

$$\begin{tikzcd}
\vec{x}(t) \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{x} \arrow[d, bend right, "V^{-1}"'] \\
\vec{\hat{x}}(t) \arrow[r, "\hat{A}=V^{-1}AV"'] \arrow[u, bend right, "V"'] & \frac{d}{dt}\vec{\hat{x}} \arrow[u, bend right, "V"']
\end{tikzcd}$$

$$\frac{d}{dt}\vec{\hat{x}} = \underbrace{V^{-1}AV}_{\hat{A}}\,\vec{\hat{x}}$$

$$\frac{d}{dt}\vec{x}(t) = \underbrace{\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}}_{A}\vec{x}(t) \qquad V = \begin{bmatrix}1 & 1\\ -j & j\end{bmatrix}$$

$$\Downarrow$$

$$\frac{d}{dt}\vec{\hat{x}}(t) = \underbrace{\begin{bmatrix}j & 0\\ 0 & -j\end{bmatrix}}_{V^{-1}AV = \Lambda = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}}\vec{\hat{x}}(t)$$

$$\vec{\hat{x}}(t) = \begin{bmatrix}\hat{x}_1(0)\,e^{jt}\\ \hat{x}_2(0)\,e^{-jt}\end{bmatrix}$$

but we also need $\hat{x}(0)$

$$\vec{\hat{x}}(t) = V^{-1}\vec{x}(t)$$

$$\vec{\hat{x}}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix}\frac{1}{2} & \frac{j}{2}\\ \frac{1}{2} & -\frac{j}{2}\end{bmatrix}\begin{bmatrix}\overset{I_L(0)=0A}{0}\\ \underset{V_c(0)=1V}{1}\end{bmatrix} = \boxed{\begin{bmatrix}\frac{j}{2}\\ -\frac{j}{2}\end{bmatrix}}$$

$$\vec{\tilde{x}}(t) = \begin{bmatrix} \frac{j}{2} e^{jt} \\ -\frac{j}{2} e^{-jt} \end{bmatrix}$$

Go back to $\vec{x}(t) = V \vec{\tilde{x}}(t) = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \begin{bmatrix} \frac{j}{2} e^{jt} \\ -\frac{j}{2} e^{-jt} \end{bmatrix}$

$$\begin{bmatrix} I_L(t) \\ V_c(t) \end{bmatrix} = \vec{x}(t) = \begin{bmatrix} \frac{j}{2} e^{jt} - \frac{j}{2} e^{-jt} \\ \frac{1}{2} e^{jt} + \frac{1}{2} e^{-jt} \end{bmatrix}$$

Euler formula (or from Taylor expansion):

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

$$\underline{x_1(t)} = \frac{j}{2} e^{jt} - \frac{j}{2} e^{-jt} = \frac{j}{2} \left(e^{jt} - e^{-jt} \right) =$$

$$= \frac{j}{2} \left(\cos(t) + j \sin(t) - (\cos(-t) + j \sin(-t)) \right)$$

$$= \frac{j}{2} \left(\cos(t) + j \sin(t) - (\cos(t) - j \sin(t)) \right)$$

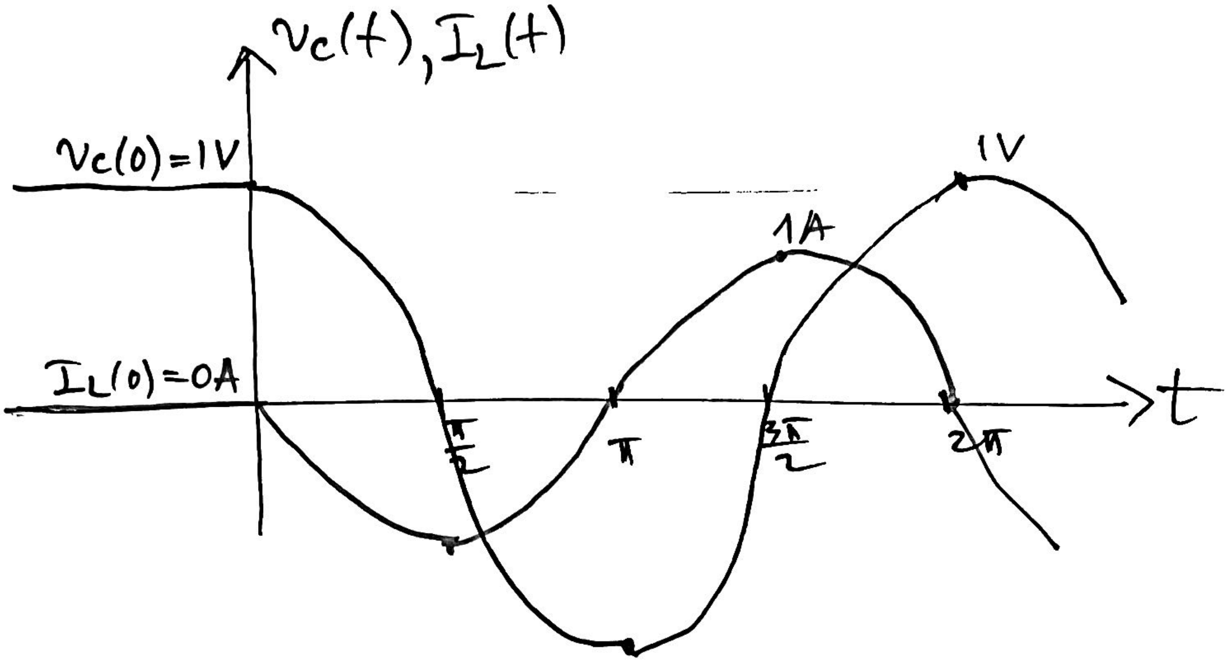
$$= \frac{j}{2} \left(\cancel{\cos(t)} + j \sin(t) - \cancel{\cos(t)} + j \sin(t) \right)$$

$$= \frac{j}{2} \cdot 2j \sin(t) = \boxed{-\sin(t)}$$

$$\underline{x_2(t)} = \frac{e^{jt} + e^{-jt}}{2} = \boxed{\cos(t)}$$

$1 \frac{\text{rad}}{s} \cdot t$ real functions

$$\vec{x}(t) = \begin{bmatrix} I_L(t) \\ V_c(t) \end{bmatrix} = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$



Note: $\lambda_{1,2} = \pm j\sqrt{\frac{1}{LC}}$

$$\vec{x}(t) = \begin{bmatrix} I_L(t) \\ v_C(t) \end{bmatrix} = \begin{bmatrix} -\sin\left(\frac{1}{\sqrt{LC}}t\right) \\ \cos\left(\frac{1}{\sqrt{LC}}t\right) \end{bmatrix} = \begin{bmatrix} -\sin(\omega_0 t) \\ \cos(\omega_0 t) \end{bmatrix}$$

$\omega_0 = \frac{1}{\sqrt{LC}}$