Lecture 9

* Filters
  * Low-pass & high-pass examples
  * Transfer functions
  * Bode plot approximations
* Cascading filters

__Objective__: To find a __steady-state__ solution of the system in response to a __sinusoidal__ inputs ($\lambda_r < 0$)

* Sinusoidal inputs allow us to transform a system of diff. eqns. into a system of linear eqns. (Phasor domain)

__Phasor analysis:__

__Step 1:__ Write independent sources (inputs) as exponentials.

$$V_{si}(t) = \hat{V}_{si} e^{j\omega_i t} + \tilde{\hat{V}}_{si} e^{-j\omega_i t}$$

$$I_{si}(t) = \hat{I}_{si} e^{j\omega_i t} + \tilde{\hat{I}}_{si} e^{-j\omega_i t}$$

$\hat{V}_{si}$ & $\hat{I}_{si}$ are phasors

__Step 2:__ Transform the circuit into the phasor domain:

(s-impedances where $s = j\omega$) $\dfrac{\hat{V}_{el}}{\hat{I}_{el}} = Z_{el}(s = j\omega)$

$$\frac{\hat{V}_R}{\hat{I}_R} = Z_R(j\omega) = R, \quad \frac{\hat{V}_c}{\hat{I}_c} = Z_c(j\omega) = \frac{1}{j\omega C}, \quad \frac{\hat{V}_L}{\hat{I}_L} = Z_L(j\omega) = j\omega L$$

step 3 Cast branch & element equations in phasor domain

KCL: $\sum_{\substack{i \text{ into} \\ \text{the node}}} \tilde{I}_i = 0$

Ohm's law:
$\tilde{V}el_i = Zel_i \cdot \tilde{I}el_i$

NVA: $\sum \frac{\hat{V}_j - \hat{V}_k}{Z_{jk}} = 0$, KVL holds as well

step 4: Solve for unknown variables: $\hat{V}el_i$, $\tilde{I}el_i$

step 5: Transform the phasor solutions $(\tilde{V}el_i, \tilde{I}el_i)$ into time-domain.

$$Vel(t) = \hat{V}el\, e^{j\omega t} + \bar{\hat{V}}el\, e^{-j\omega t}$$

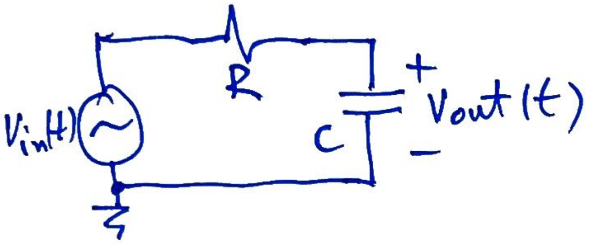
$$Iel(t) = \tilde{I}el\, e^{j\omega t} + \bar{\tilde{I}}el\, e^{-j\omega t}$$

$$\Downarrow$$

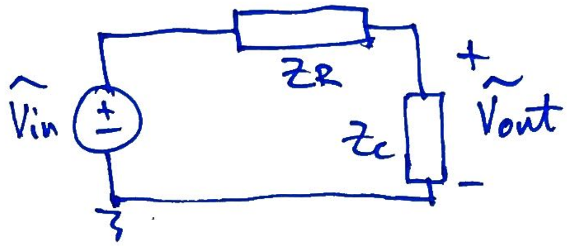
$$Vel(t) = 2|\hat{V}el| \cos(\omega t + \measuredangle \hat{V}el)$$

$$Iel(t) = 2|\tilde{I}el| \cos(\omega t + \measuredangle \tilde{I}el)$$

Example 1 continued: Low-pass filter



$\Rightarrow$



$V_{in}(t) = V_{in}\cos(\omega_{in} t + \phi)$

time-domain view

phasor-domain

$\underline{\tilde{V}_{out} = ?}$:
$$\underbrace{\frac{\tilde{V}_{out}}{\tilde{V}_{in}}}_{H_{LP}(j\omega)\text{ - transfer function}} = \frac{Z_C}{Z_C + Z_R}$$

$$H_{LP}(j\omega) = \frac{Z_C}{Z_C+Z_R} = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\frac{\omega}{\omega_0}}$$

$$\boxed{H_{LP}(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_0}}}$$

$\omega_0 = \frac{1}{RC}$

"cut-off" frequency

$\tilde{V}_{out} = H_{LP}(j\omega_{in}) \cdot \tilde{V}_{in}$

$$|H_{LP}(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}} \xrightarrow{\omega \gg \omega_0} 0$$

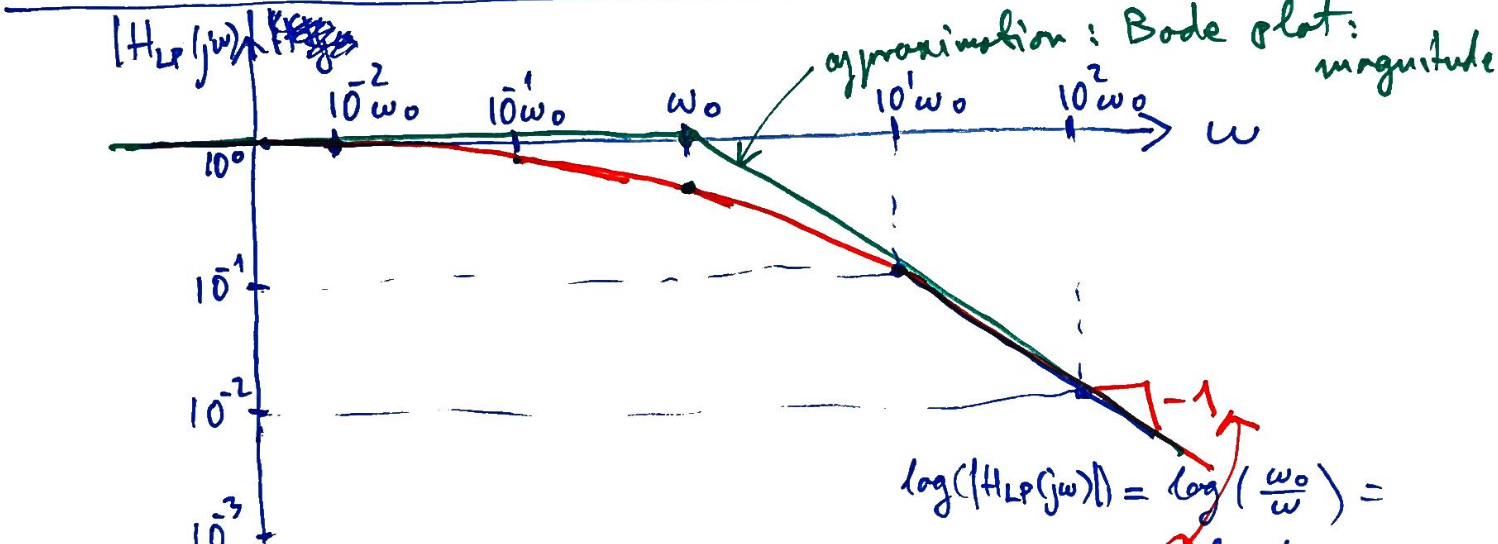
$$\xrightarrow{\omega \ll \omega_0} 1$$

$$\measuredangle H_{LP}(j\omega) = -\operatorname{atan2}\left(\frac{\omega}{\omega_0}, 1\right) \xrightarrow{\omega \gg \omega_0} -\frac{\pi}{2} \quad (-90^\circ)$$

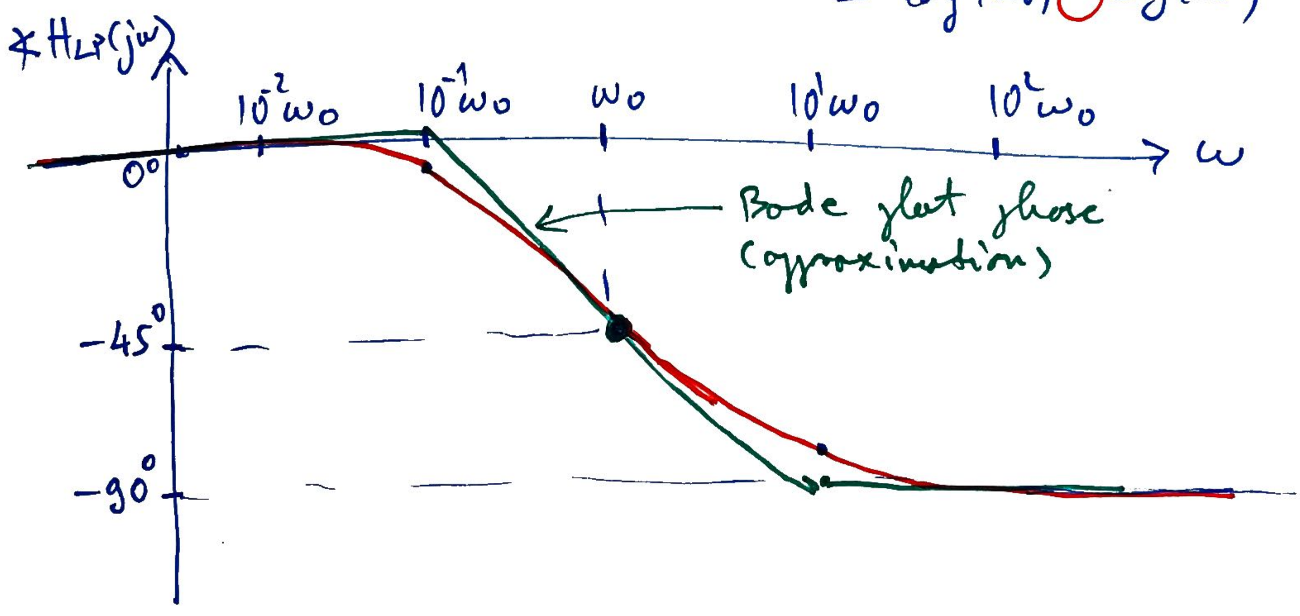
$$\xrightarrow{\omega \ll \omega_0} 0 \quad (0^\circ)$$

(e4)

ω	$H_{LP}(j\omega)$	$\lvert H_{LP}(j\omega)\rvert$	$\measuredangle H_{LP}(j\omega)$
$\omega \ll \omega_0$	$\simeq 1$	1	$\sim 0^\circ$
$0.1\omega_0$	$\frac{1}{1+j0.1}$	0.995	-6°
ω_0	$\frac{1}{1+j}$	$\frac{1}{\sqrt{2}} \simeq 0.71$	-45°
$10\omega_0$	$\frac{1}{1+j10}$	0.1	-84°
$\omega \gg \omega_0$	$-j\frac{\omega_0}{\omega}$	$\frac{\omega_0}{\omega}$	-90°



$$\log(|H_{LP}(j\omega)|) = \log\left(\frac{\omega_0}{\omega}\right) =$$
$$= \log(\omega_0) - \log(\omega)$$



$$V_{out}(t) = \tilde{V}_{out} e^{j\omega t} + \hat{\tilde{V}}_{out} e^{-j\omega t}$$

$$= 2 |\tilde{V}_{out}| \cos(\omega t + \measuredangle \tilde{V}_{out}) \qquad (1)$$

$$\tilde{V}_{out} = H_{LP}(j\omega) \cdot \tilde{V}_{in}$$

$$|\tilde{V}_{out}| e^{j\measuredangle \tilde{V}_{out}} = |H_{LP}(j\omega)| \cdot e^{j\measuredangle H_{LP}(j\omega)} \cdot |\tilde{V}_{in}| \cdot e^{j\measuredangle \tilde{V}_{in}}$$

$$|\tilde{V}_{out}| e^{j\measuredangle \tilde{V}_{out}} = |H_{LP}(j\omega)| \cdot |\tilde{V}_{in}| \, e^{j(\measuredangle H_{LP}(j\omega) + \measuredangle \tilde{V}_{in})}$$

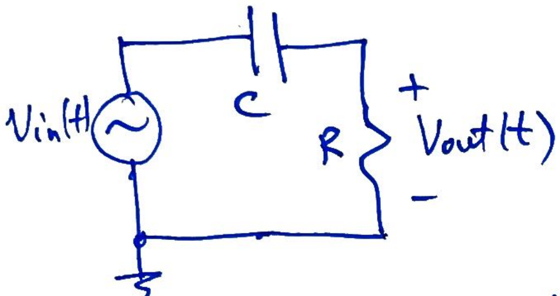
$$\boxed{\begin{aligned} |\tilde{V}_{out}| &= |H_{LP}(j\omega)| \, |\tilde{V}_{in}| \quad @ \; \omega = \omega_{in} \\ \measuredangle \tilde{V}_{out} &= \measuredangle \tilde{V}_{in} + \measuredangle H_{LP}(j\omega) \quad @ \; \omega = \omega_{in} \end{aligned}}$$

$$V_{out}(t) = \Big| 2 |H_{LP}(j\omega)| \cdot |\tilde{V}_{in}| \cos(\omega t + \measuredangle \tilde{V}_{in} + \measuredangle H_{LP}(j\omega)) \Big._{\omega = \omega_{in}}$$

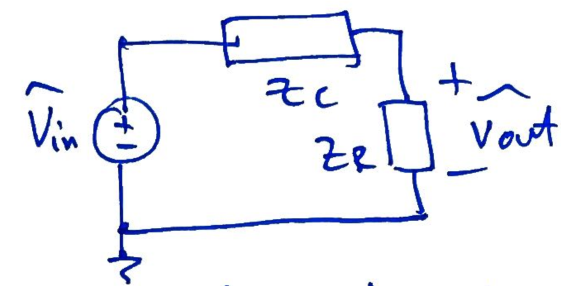
for input $v_{in}(t) = V_{in} \cos(\omega_{in} t + \phi)$

$$\boxed{V_{out}(t) = 2 |H_{LP}(j\omega_{in})| \, |\tilde{V}_{in}| \cos(\omega_{in} t + \cancel{\phi} + \measuredangle H_{LP}(j\omega_{in}))}$$

Example 2: High-pass filter



$\Rightarrow$



$V_{in}(t) = V_{in}\cos(\omega_{in} t + \phi)$

time-domain

phasor-domain

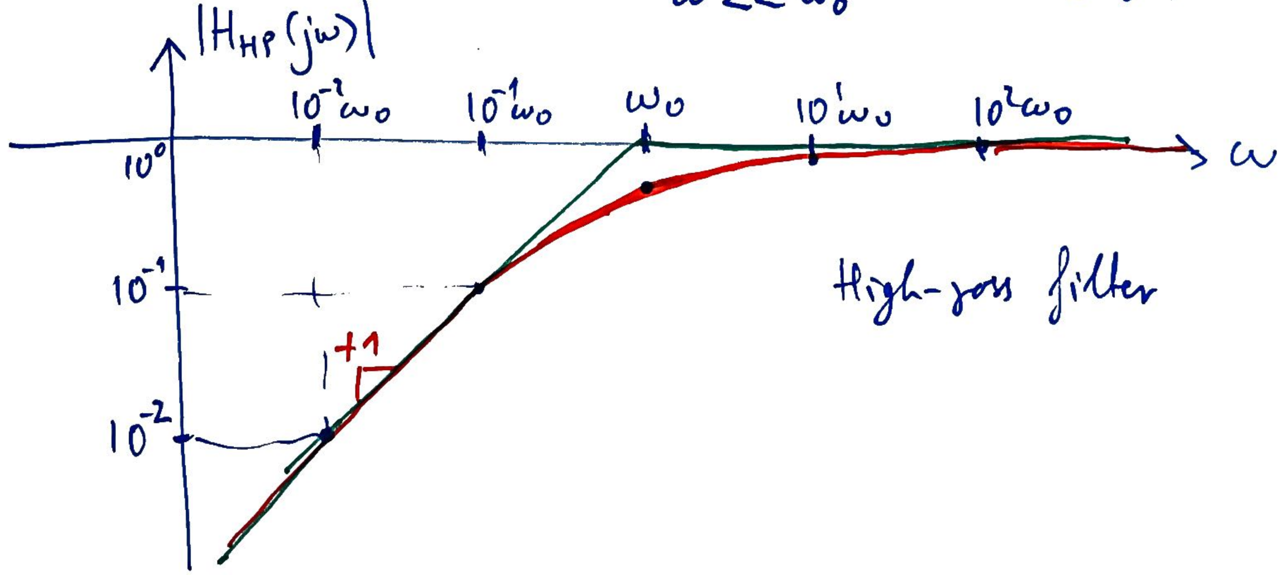
$$H_{HP}(j\omega) = \frac{\hat{V}_{out}}{\hat{V}_{in}} = \frac{Z_R}{Z_R + Z_C} = \frac{R}{R + \frac{1}{j\omega C}}$$

$$H_{HP}(j\omega) = \frac{j\omega RC}{1 + j\omega RC}, \quad \omega_0 = \frac{1}{RC}$$ "cut-off" frequency

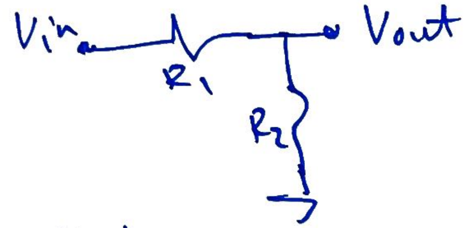
$$\boxed{H_{HP}(j\omega) = \frac{j\frac{\omega}{\omega_0}}{1 + j\frac{\omega}{\omega_0}} = \frac{1}{1 - j\frac{\omega_0}{\omega}}}$$

$$|H_{HP}(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega_0}{\omega}\right)^2}} \xrightarrow[\omega \gg \omega_0]{} 1$$

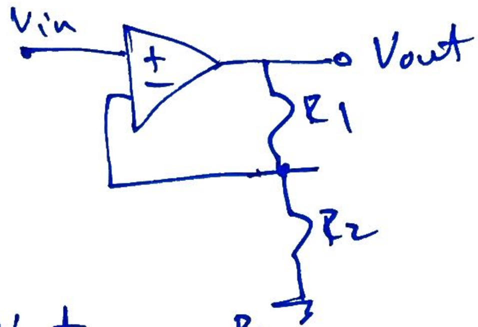
$$\xrightarrow[\omega \ll \omega_0]{} 0 \quad \left(\frac{\omega}{\omega_0}\right)$$



In 16A:



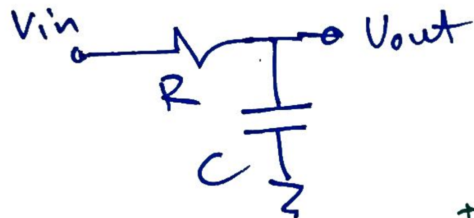
$$\frac{V_{out}}{V_{in}} = \frac{R_2}{R_1+R_2}$$



$$\frac{V_{out}}{V_{in}} = 1 + \frac{R_1}{R_2}$$

"Operators": Transfer functions

16B:



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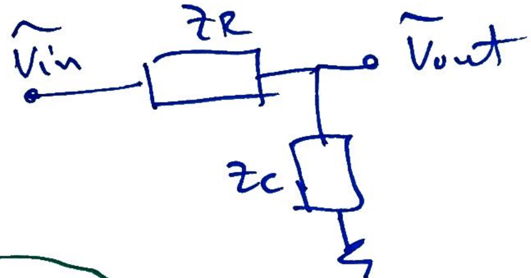
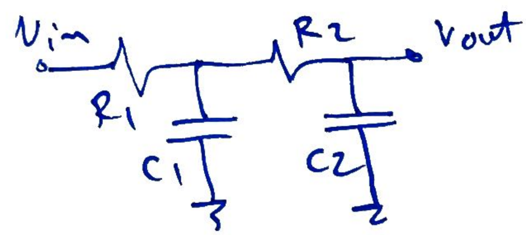


$$\boxed{\frac{\tilde{V}_{out}}{\tilde{V}_{in}} = H(j\omega)} = \frac{Z_C}{Z_C+Z_R}$$

transfer function

How do we cascade circuits in a tractable way to build more complex transfer functions?

Circuit blocks should not "load" (i.e. take current from) each other in order to preserve the transfer function.



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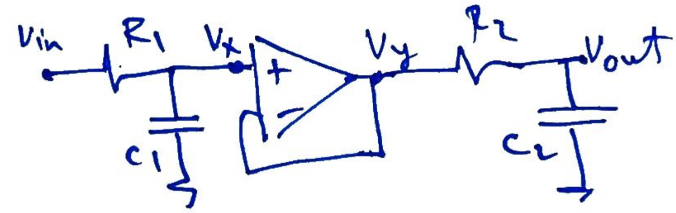

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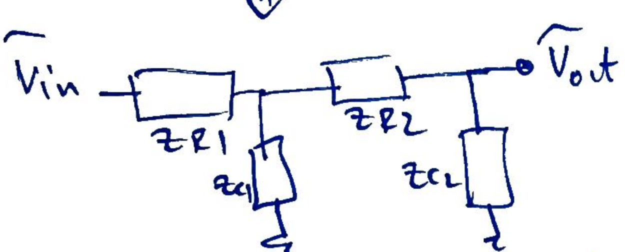

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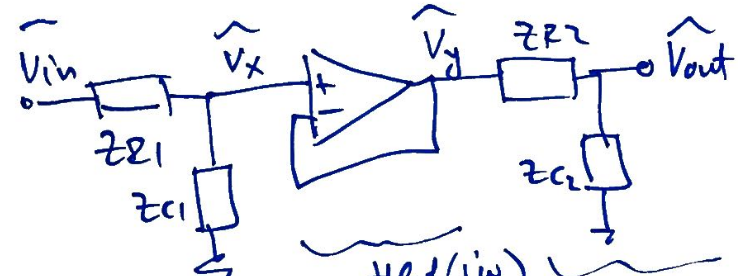

$H_1(j\omega)$ $\quad$ $H_{buf}(j\omega)$ $\quad$ $H_2(j\omega)$

Need to solve the full ckt to find $\frac{\hat{V}_{out}}{\hat{V}_{in}} = H(j\omega)$

$$H_1(j\omega) = \frac{\hat{V}_x}{\hat{V}_{in}}$$

$$H_{buf}(j\omega) = \frac{\hat{V}_y}{\hat{V}_x} = 1$$

$$H_2(j\omega) = \frac{\hat{V}_{out}}{\hat{V}_y}$$

$$H(j\omega) = \frac{\hat{V}_{out}}{\hat{V}_{in}} = \underbrace{\frac{\hat{V}_{out}}{\hat{V}_y}}_{H_2(j\omega)} \cdot \underbrace{\frac{\hat{V}_y}{\hat{V}_x}}_{H_{buf}(j\omega)} \cdot \underbrace{\frac{\hat{V}_x}{\hat{V}_{in}}}_{H_1(j\omega)}$$

$$H_1(j\omega) = \frac{Z_{C1}}{Z_{C1} + Z_{R1}}$$

$$H_2(j\omega) = \frac{Z_{C2}}{Z_{C2} + Z_{R2}}$$

$$H_1(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_{01}}}, \quad H_2(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_{02}}}$$

$$\omega_{01} = \frac{1}{R_1 C_1} \qquad \omega_{02} = \frac{1}{R_2 C_2}$$

$$H(j\omega) = H_1(j\omega)\,\overset{1}{\cancel{H_{buf}(j\omega)}}\,H_2(j\omega)$$

$$= \frac{1}{1 + j\frac{\omega}{\omega_{01}}} \cdot \frac{1}{1 + j\frac{\omega}{\omega_{02}}}$$

<u>In general:</u>

$$\hat{V}_{in} \to \underbrace{\boxed{H_1(j\omega)} \to \cdots \to \boxed{H_n(j\omega)}}_{H(j\omega)} \to \hat{V}_{out}$$

$$H(j\omega) = \frac{\hat{V}_{out}}{\hat{V}_{in}} = H_1(j\omega) \cdots H_n(j\omega) \quad , \quad H_i(j\omega) = |H_i(j\omega)|\, e^{j\measuredangle H_i(j\omega)}$$

$$H(j\omega) = \underbrace{|H_1(j\omega)| \cdots |H_n(j\omega)|}_{|H(j\omega)|}\, e^{j\overbrace{\left(\measuredangle H_1(j\omega) + \cdots + \measuredangle H_n(j\omega)\right)}}$$

$$ \measuredangle H(j\omega)$$

time domain : $\left(V_{in}(t) = V_{in}\cos(\omega_{in} t + \phi)\right)$

$$V_{out}(t) = |H(j\omega = \omega_{in})|\, 2\, |\hat{V}_{in}| \cos(\omega_{in} t + \measuredangle \hat{V}_{in} + \measuredangle H(j\omega_{in}))$$

$$V_{out}(t) = |H(j\omega = \omega_{in})|\, V_{in} \cos(\omega_{in} t + \phi + \measuredangle H(j\omega_{in}))$$