Note 8: Bode Plots

Overview

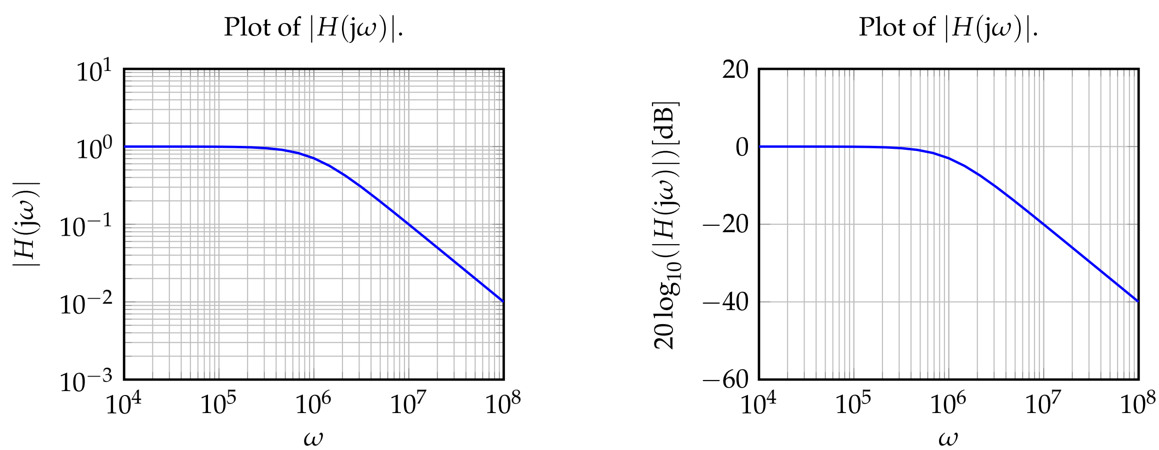
In this note, we introduce the concept of Bode Plots, which are piecewise-linear approximations to transfer function plots. These will not only prove useful in plotting a filter's frequency response by hand (as opposed to with a computer or plotting software), but will also help us better understand the behavior of cascaded filters. This note will likely feel like review from the Transfer Function Plotting Note, and much of the later content is optional. Bode Plots are quite useful for performing filter design by hand quickly for various applications.

This note will present 2 key ideas, which build on what you've learned about Transfer Functions.

- Linear Approximations: Linear approximations applied to a transfer function plot define a Bode Plot, and this is where the key benefit comes in with respect to design.
- $y-$axis scaling: For the magnitude plot, we will define a transformation of the axes from the "standard log scale" (powers of 10) into the decibel scale. This change is to make some of the graphical aspects of Bode Plot compositions easier.

1 Bode Plots

First, we will handle the modification in the scale of the magnitude plot's axis from a logarithmic scale $(10^{-1}, 10^0, 10^1, 10^2, ...)$ to an adjusted linear, decibel scale $(-20, 0, 20, 40, ...)$. That is, instead of plotting $|H(j\omega)|$ vs. ω where the y-axis is on a *logarithmic* scale, we plot $20\log_{10}(|H(j\omega)|)$ vs. ω instead, and now the y-axis is on a *linear* scale. This linear scale is referred to as the *decibel scale* because of the multiplication by 20.[1]



[1] The reason the constant 20 is used is explained in the previous note; it's an artifact of convention.

1.1 Low-Pass Filter

As a reminder, our low-pass filter has the following form:

$$H_{\text{LP}}(\text{j}\omega) = \frac{1}{1 + \text{j}\omega/\omega_c} \tag{1}$$

1.1.1 Low-Pass Filter: Magnitude Bode Plot

In addition to plotting the magnitude of the frequency response (that is, the exact transfer function magnitude), we would like to develop a piecewise-linear approximation as well. A lot of the mathematical groundwork has been laid in Note 7, where we discuss the different regions of the plot ($\omega \ll \omega_c$, $\omega = \omega_c$, $\omega \gg \omega_c$) and convey the connection between the logarithm properties and slopes of the lines. Here, we will supplement this with a graphical approach.

There are 2 distinct regions of the magnitude plot to examine from the perspective of piecewise-linear segments. At frequencies much below the cutoff $\omega \ll \omega_c$, the magnitude plot is effectively a horizontal line. So, we can draw that with a dashed segment. For frequencies much larger than cutoff $\omega \gg \omega_c$, we have a line with a decreasing slope (of -1). We similarly draw this asymptote, dashed. At this point, we have a plot as shown in fig. 1. Note the use of 2 equivalent $y-$axes; on the left is the log scale that we've used in prior notes, and on the right is the decibel scale introduced in this note.

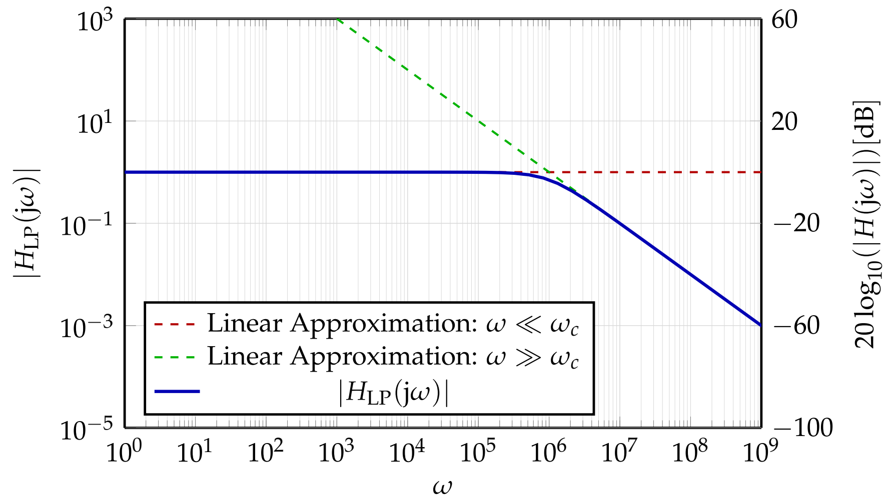


Figure 1: Low-pass filter magnitude plot, with 2 asymptotes drawn (dashed).

Now, once we plot these both, there is a point of conflict in the middle, right around ω_c. In this region, we will join the two models at a point, and pick the corresponding model for a given region (horizontal for $\omega < \omega_c$, sloped for $\omega > \omega_c$.) This means our final Bode Plot for the magnitude of a low-pass filter is as shown in fig. 2.

Why do we pick this approach? Well, let's first outline the problem in a bit more detail. Around ω_c, the sloped line claims that the magnitude at frequencies lower than ω_c should keep increasing, whereas the horizontal line in that region claims the magnitude is straight. Similarly, the horizontal line claims that the magnitude at frequencies higher than ω_c should stay constant, whereas the sloped line in that region claims the magnitude is decreasing. How do we resolve this difference?

What we do here is default to unilaterally picking the model that is better for a given region. That's why we abruptly transition from one regime to the other; at $\omega_c - \epsilon$ for some small ϵ, the straight line is better, so we pick that curve. At $\omega_c + \epsilon$, we're now closer to the sloped model, so we start to slope down. This is to

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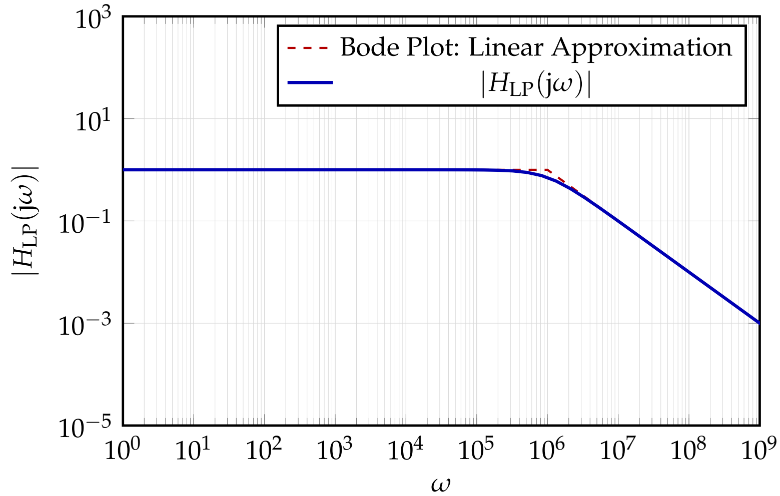

Figure 2: Low-pass filter magnitude plot.

maintain simplicity while staying true (within bounded error) to the actual plot, which we know the shape of.

Can we be more precise about this bound on the error? It is a factor of $\frac{1}{\sqrt{2}}$, and this occurs right at the cutoff frequency. In other words, using the Bode Plot approximation will be inaccurate by at most 30%, and this level of inaccuracy arises at frequencies ω that are "close" to the cutoff frequency ω_c. It is a common rule of thumb to say that Bode Plot is correct for frequencies at least a factor of 10 away from cutoff; for anything within $0.1\omega_c$ to $10\omega_c$, it's safest to evaluate the transfer function exactly when possible.) The farther away we are from ω_c, the better the approximation will be, as seen in the plot.

1.1.2 Low-Pass Filter: Phase Bode Plot

Now, let's perform the same approximation process for the phase plot of a low-pass filter's transfer function (to arrive at the Bode Plot for the phase of a low-pass filter). In this case, there are 3 regions to examine. At frequencies much below the cutoff $\omega \ll \omega_c$, the phase plot is effectively a horizontal line with value $0°$. So, we can draw that with a dashed segment. For frequencies much larger than cutoff $\omega \gg \omega_c$, we have a horizontal line with value $-90°$. Finally, there is the middle transition region during which we curve from $0°$ at low frequencies down to $-90°$ at high frequencies. For this region too, we will use a line.

However, there's a point of subtlety here; how do we choose the *slope* of this line? The piecewise-linear approximation in this linear region should probably go through $-45°$ at $\omega = \omega_c$, since this is exactly correct based on the true transfer function plot. But, where will this line intersect the other two horizontal lines? Should it be at $5\omega_c$ and $\frac{\omega_c}{5}$? $15\omega_c$ and $\frac{\omega_c}{15}$? It isn't immediately clear what's best. For design simplicity, we will choose to model the region between $10\omega_c$ and $\frac{\omega_c}{10}$ with this line. This is consistent with our reasoning and prior approach; even back in the Filters note when we began analyzing transfer functions, our tables of values used $0.1\omega_c$ and $10\omega_c$, so the concept is hopefully faimilar.

Why a factor of 10 exactly in the first place? There are several valid reasons, but the most important one is *design simplicity*. The log scale for the frequency axis is naturally divided into increments of 10. This means that even if our cutoff frequency for some filter isn't at a clean multiple of 10 (for example, suppose $\omega_c = 6 \times 10^{6}\,\frac{\mathrm{rad}}{\mathrm{s}}$), we can easily sketch[2] the phase plot to be horizontal until $6 \times 10^{5}\,\frac{\mathrm{rad}}{\mathrm{s}}$, sloping down to connect $0°$ and $-90°$ between $6 \times 10^{5}\,\frac{\mathrm{rad}}{\mathrm{s}}$ and $6 \times 10^{7}\,\frac{\mathrm{rad}}{\mathrm{s}}$, and then again horizontal after $6 \times 10^{7}\,\frac{\mathrm{rad}}{\mathrm{s}}$. We cannot do this easily with alternatives (like factors of 7 or 12.)

[2] Keep in mind, these are for *hand*-drawings when we do design!

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At this point, we have a plot as shown in fig. 3, with the 3 asymptotes drawn. The $y-$axis is already linear in degrees[3], so we don't have a decibel scale or anything.

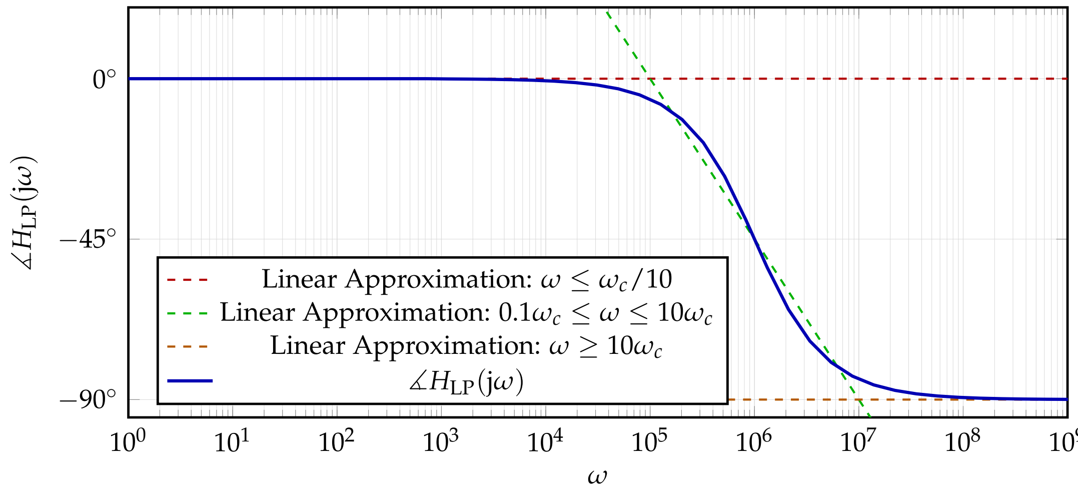

Figure 3: Low-pass filter phase plot, with 3 asymptotes drawn (dashed).

Once we join the asymptotes in their corresponding regimes, as discussed above, we arrive at fig. 4 for the Bode Plot of a low-pass filter's phase.

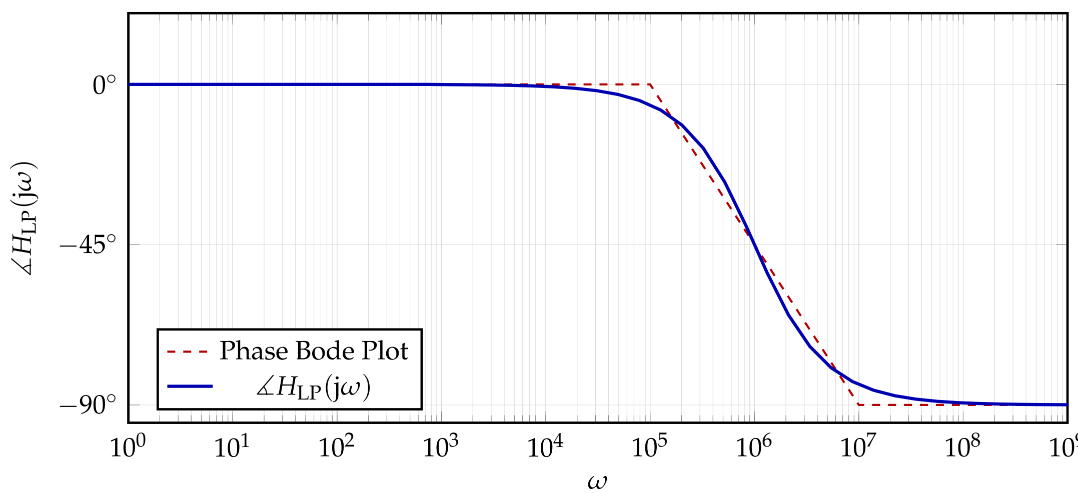

Figure 4: Low-pass filter phase plot.

1.2 High-Pass Filter

We can similarly analyze our generalized high-pass filter model:

$$H_{\text{HP}}(\text{j}\omega) = \frac{\text{j}\omega/\omega_c}{1 + \text{j}\omega/\omega_c} \tag{2}$$

All of the same reasoning as for the low-pass filter holds here but in "reverse," so for succintness, we will directly draw the Bode Plots for the high pass filter's magnitude (fig. 5) and phase (fig. 6) assuming $\omega_c = 10^6$.

[3] Such plots could easily be formulated in terms of radians too.

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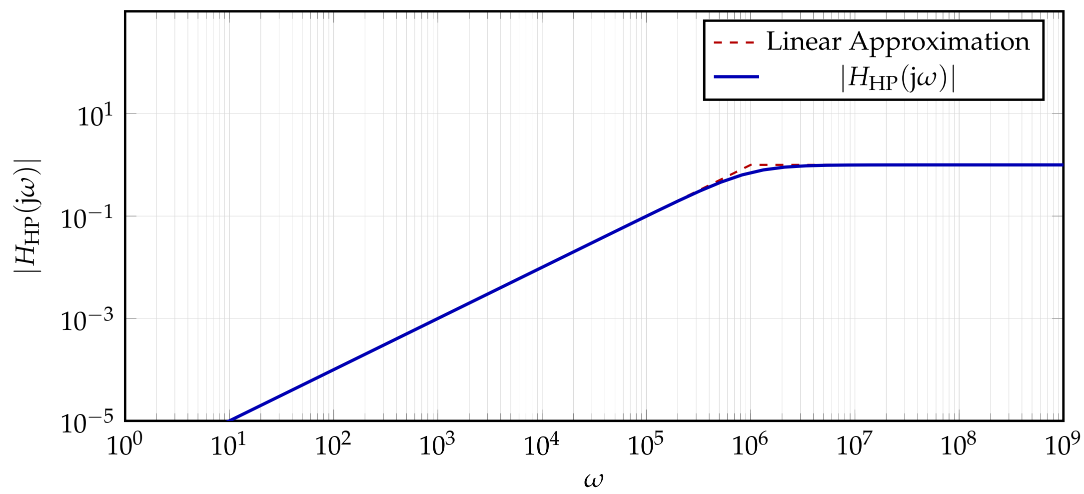

Figure 5: High-pass filter magnitude plot.

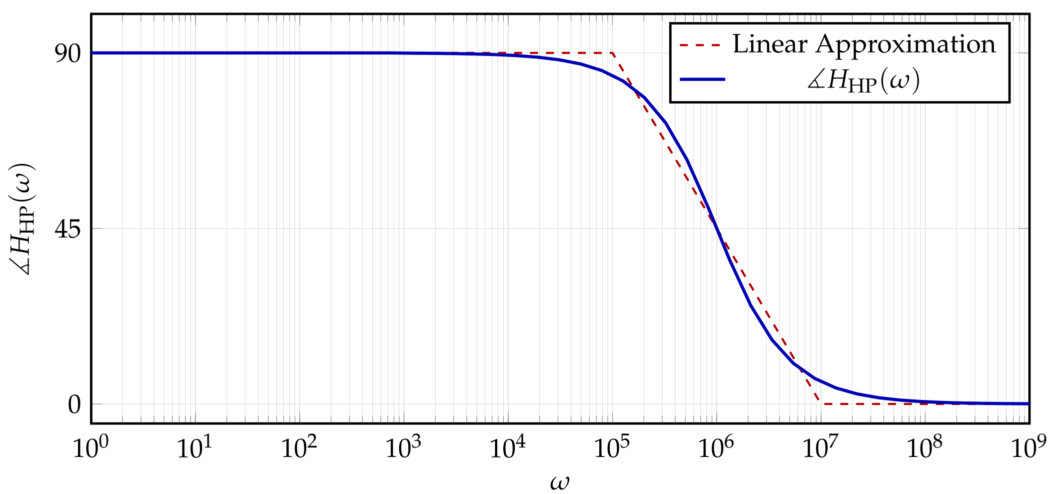

Figure 6: High-pass filter phase plot.

2 Plotting Linear Approximations of General Transfer Functions

2.1 Composing Transfer Functions

For two transfer functions $H_1(\mathrm{j}\omega)$ and $H_2(\mathrm{j}\omega)$, if $H(\mathrm{j}\omega) = H_1(\mathrm{j}\omega) \cdot H_2(\mathrm{j}\omega)$,

$$\log|H(\mathrm{j}\omega)| = \log|H_1(\mathrm{j}\omega) \cdot H_2(\mathrm{j}\omega)| = \log|H_1(\mathrm{j}\omega)| + \log|H_2(\mathrm{j}\omega)| \tag{3}$$

$$\measuredangle H(\mathrm{j}\omega) = \measuredangle(H_1(\mathrm{j}\omega) \cdot H_2(\mathrm{j}\omega)) = \measuredangle H_1(\mathrm{j}\omega) + \measuredangle H_2(\mathrm{j}\omega) \tag{4}$$

As a consequence, when plotting $|H(\mathrm{j}\omega)|$ on a decibel scale plot, we can simply plot $|H_1(\mathrm{j}\omega)|$ and $|H_2(\mathrm{j}\omega)|$ and add them (unlike with a log-log plot where we have to pointwise-multiply transfer functions of each stage to arrive at an accurate overall transfer function). This holds too for the transfer functions' Bode plots.

This is all that you are expected to understand in scope for 16B.

3 Rational Transfer Functions [Optional: not in 16B scope]

When we write the transfer function of an arbitrary circuit involving linear circuit elements, it can always be rearranged to take the following form, called a "rational transfer function:"

$$H(\mathrm{j}\omega) = K \cdot \frac{N(\mathrm{j}\omega)}{D(\mathrm{j}\omega)} \tag{5}$$

where the numerator $N(x)$ is a polynomial and so is the denominator $D(x)$.

The reasons for this rational form always emerging are themselves interesting, and related to the problem on the HW where you work with a phasor style derivation with a system of differential equations expressed in vector-matrix form. Basically, it is a consequence of how determinants and matrix inverses behave. The full reasons for this are just outside of mathematical accessibility given 16B's mathematical maturity assumptions and so you can view this as an empirical observation for now. But this is partially why this entire section is outside of 16B's scope.

We like to factor the numerator and denominator so that they become easier to work with and plot:

$$H(\mathrm{j}\omega) = K \cdot \frac{N(\mathrm{j}\omega)}{D(\mathrm{j}\omega)} = K \frac{(\mathrm{j}\omega)^{N_{z0}}\left(1+\mathrm{j}\frac{\omega}{\omega_{z1}}\right)\left(1+\mathrm{j}\frac{\omega}{\omega_{z2}}\right)\cdots\left(1+\mathrm{j}\frac{\omega}{\omega_{zn}}\right)}{(\mathrm{j}\omega)^{N_{p0}}\left(1+\mathrm{j}\frac{\omega}{\omega_{p1}}\right)\left(1+\mathrm{j}\frac{\omega}{\omega_{p2}}\right)\cdots\left(1+\mathrm{j}\frac{\omega}{\omega_{pm}}\right)}. \tag{6}$$

The above is a consequence of the Fundamental Theorem of Algebra which asserts that all polynomials with complex coefficients can be factored into monomials — i.e. all complex polynomials of degree d have exactly d roots, if one counts repetitions. This is a theorem that you have probably seen asserted, but is usually only properly proved in upper-division mathematics courses — for example, complex analysis: Math 185.

The above factorization is interesting because it says that no matter what, we can think of a transfer function as though it were a composition of elementary filters connected by unity-gain buffers. Mathematically, because polynomials might in principle have no constant terms, we need to deal with two more new objects that don't correspond to the simple low-pass and high-pass filters that we have seen so far.

To summarize the components, each transfer function is the product of constant gain K, one or more "origin poles" ($(\mathrm{j}\omega)$ in the denominator) or "origin zeros" ($(\mathrm{j}\omega)$ in the numerator) — these are the two new things — and one or more "poles" ($\left(1+\mathrm{j}\frac{\omega}{\omega_{pi}}\right)$ in the denominator) or "zeros" ($\left(1+\mathrm{j}\frac{\omega}{\omega_{zi}}\right)$ in the numerator). This specific terminology regarding poles and zeros is not in scope for this class; it will come up in future circuits and controls classes and is borrowed from complex analysis.

Here, we define the constants ω_z as "zeros" and ω_p as "poles."" The zeros are the roots of $N(\mathrm{j}\omega)$ while poles are the roots of $D(\mathrm{j}\omega)$. [4]

Using our rules for composing Bode plots, we know how to decompose our so-called general rational transfer function in terms of its magnitude and phase:

$$|H(\mathrm{j}\omega)| = |K| \cdot \frac{|\mathrm{j}\omega|^{N_{z0}} \cdot \left|1+\mathrm{j}\frac{\omega}{\omega_{z1}}\right| \cdots \left|1+\mathrm{j}\frac{\omega}{\omega_{zn}}\right|}{|\mathrm{j}\omega|^{N_{p0}} \cdot \left|1+\mathrm{j}\frac{\omega}{\omega_{p1}}\right| \cdots \left|1+\mathrm{j}\frac{\omega}{\omega_{pm}}\right|} \tag{7}$$

$$= |K| \cdot |\omega|^{N_{z0}-N_{p0}} \cdot \frac{\sqrt{1+\frac{\omega^2}{\omega_{z1}^2}} \cdots \sqrt{1+\frac{\omega^2}{\omega_{zn}^2}}}{\sqrt{1+\frac{\omega^2}{\omega_{p1}^2}} \cdots \sqrt{1+\frac{\omega^2}{\omega_{pm}^2}}} \tag{8}$$

[4] Technically if $s = \mathrm{j}\omega$, then the roots of $N(s)$ and $D(s)$ are $-\omega_z$ and $-\omega_p$. However, when plotting Bode plots, we refer to ω_z and ω_p as the zero and pole frequencies.

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$$\measuredangle H(\mathrm{j}\omega) = \measuredangle(K) + \measuredangle(\mathrm{j}\omega)^{N_{z0}} + \sum_{i=1}^{n} \measuredangle\left(1 + \mathrm{j}\frac{\omega}{\omega_{zi}}\right) - \measuredangle(\mathrm{j}\omega)^{N_{p0}} - \sum_{i=1}^{m} \measuredangle\left(1 + \mathrm{j}\frac{\omega}{\omega_{pi}}\right) \tag{9}$$

$$= \measuredangle(K) + (N_{z0} - N_{p0})\measuredangle(j) + \sum_{i=1}^{n} \mathrm{atan2}\left(\frac{\omega}{\omega_{zi}}, 1\right) - \sum_{i=1}^{m} \mathrm{atan2}\left(\frac{\omega}{\omega_{pi}}, 1\right). \tag{10}$$

Now, we have simplified as much as we can, generally. We could now convert j into either $e^{j\frac{\pi}{2}}$ or $45°$, depending on whether we're using radians or degrees for phase.

3.1 Poles, Zeros, and Constants [Optional]

3.1.1 Simple Pole, Simple Zero

The notion of a **pole** and **zero** frequency is a generalization of the term cutoff frequency.[5]Let's first look back at a plot of our RC low-pass filter, which has the following form (except we've substituted the more general ω_p for ω_c):

$$H_P(\mathrm{j}\omega) = \frac{1}{1 + \mathrm{j}\omega/\omega_p} \tag{11}$$

In what follows, pay special attention to the Linear Approximations! When drawing Bode plots, we claim that the plot drops off with a slope of 1 after a pole ω_p. Suppose our transfer function has a simple pole at $\omega_p = 10^6$. Then magnitude plot has a familiar shape as in fig. 7 (resembling a low-pass filter's transfer function!).

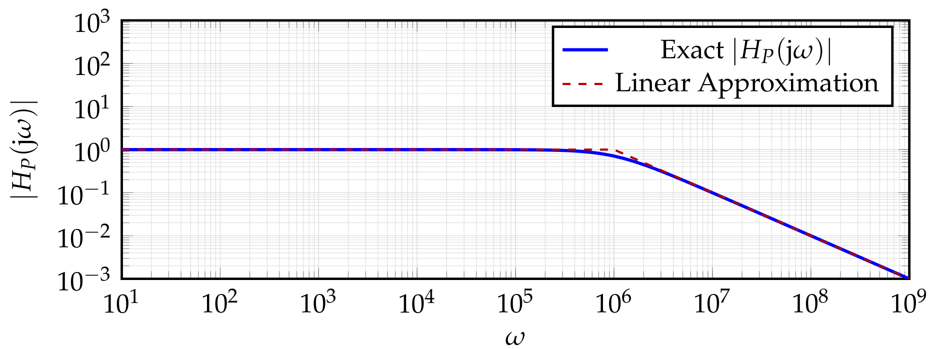

Figure 7: Simple Pole, Magnitude Plot.

We can look at the phase plot in fig. 8 as well:

Now let's take a look at a simple zero.

$$H_Z(\mathrm{j}\omega) = 1 + \mathrm{j}\omega/\omega_z \tag{12}$$

We see that this Magnitude Bode plot in fig. 9 rises with a slope of 1 after the zero at ω_z. Suppose $\omega_z = 10^6$ also.

[5]How is it more general? As an example, in all our previous plots and transfer functions, our magnitude has always *dropped* before or after the cutoff frequency relative to the passband; for a zero, it will rise.

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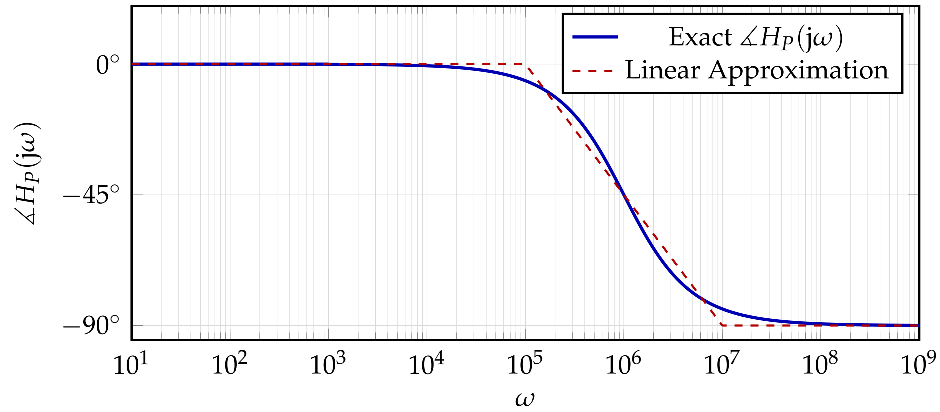

Figure 8: Simple Pole, Phase Plot.

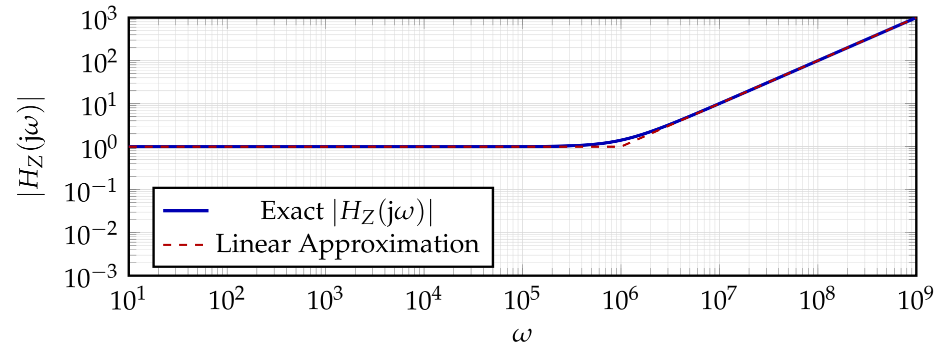

Figure 9: Simple Zero, Magnitude Plot.

The phase plot is in fig. 10.

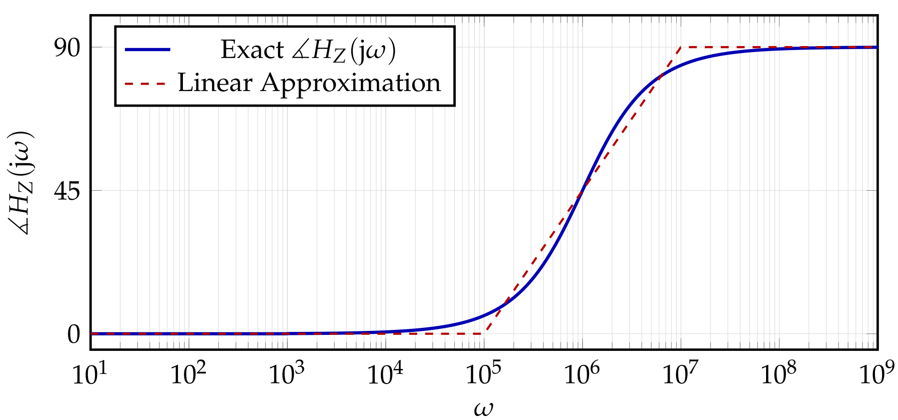

Figure 10: Simple Zero, Phase Plot.

3.1.2 Pole/Zero at the Origin

To plot a pole at the origin (as in fig. 11), recall that $H(j\omega) = \frac{1}{j\omega}$ has magnitude ω and phase $-90°$.[6] If our transfer function has a pole at the origin, it will start off with a slope of -1. The phase of a pole at the origin is $-90°$ at all frequencies.

[6]For this subsection and the next, our linear approximations are actually exactly correct.

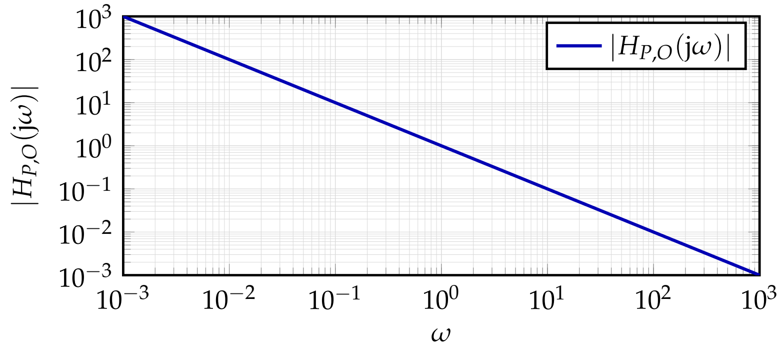

Figure 11: Origin Pole, Magnitude Plot.

We plot a zero at the origin in fig. 12, recall that $H(j\omega) = j\omega$ has magnitude ω and phase $90°$. If our transfer function has a zero at the origin, it will start off with a slope of 1. The phase of a zero at the origin is $90°$ at all frequencies.

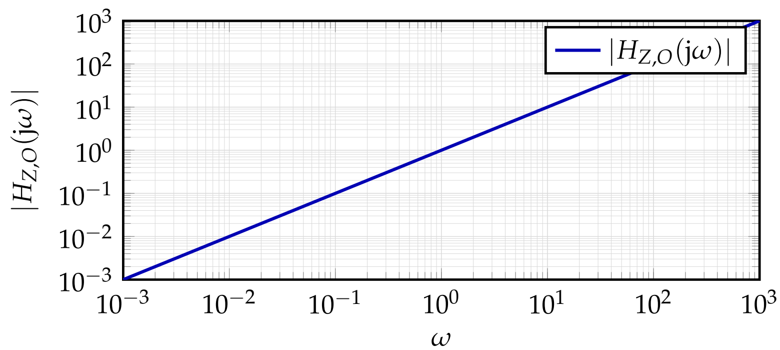

Figure 12: Origin Zero, Magnitude Plot.

3.1.3 Constant Terms

Lastly, we show the plot of a constant $K = 100$ in fig. 13. As expected, the plot remains constant. This implies that multiplication by K will shift up the entire bode plot up by K. Note that positive constants have a constant phase of $0°$ at all frequencies, while negative constants have a constant phase of $180° \equiv -180°$ at all frequencies.

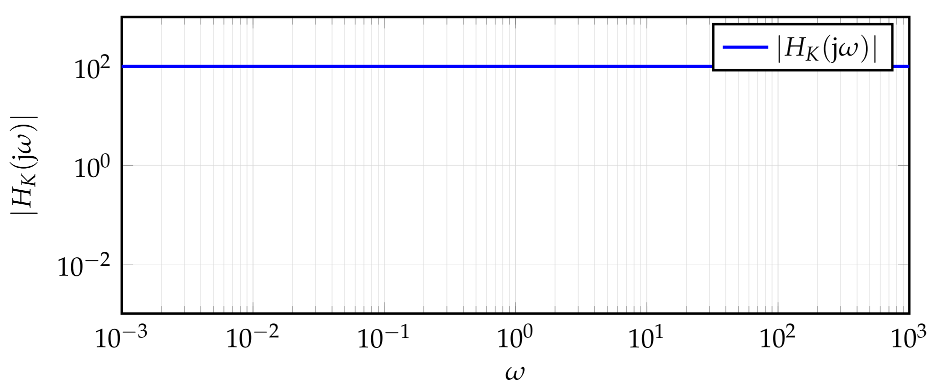

Figure 13: Constant Gain Term, Magnitude Plot.

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3.2 Bode Plot: Complicated Examples [Optional]

We have previously seen examples of how to compute the transfer function plot of a bandpass filter and for an n^{th} order low-pass filter. At this point, see if you can go back and compose them yourself with the linear approximations presented above. See if you get the same results!

Transfer Function Example

Now let's take a look at the Bode plot of a new transfer function in eq. (13).

$$H_T(\mathrm{j}\omega) = 100\frac{(1+\mathrm{j}\omega)}{(\mathrm{j}\omega)^2 + 1010(\mathrm{j}\omega) + 10^4} \tag{13}$$

Our first step is to factor this into its rational transfer function form:

$$H_T(\mathrm{j}\omega) = 0.01\frac{(1+\mathrm{j}\omega)}{(1+\frac{\mathrm{j}\omega}{10})(1+\frac{\mathrm{j}\omega}{10^3})} \tag{14}$$

With $H_T(\mathrm{j}\omega)$ in its rational form, we see that $K = 0.01, \omega_z = 1, \omega_{p1} = 10, \omega_{p2} = 10^3$. In fig. 14 is a magnitude plot of each consituent component (following the building-block rules presented above), and the multiplication of all of these provides $|H_T(\mathrm{j}\omega)|$. The linear approximations are omitted to keep the plot legible, but the approximate result will very closely match the exact one.

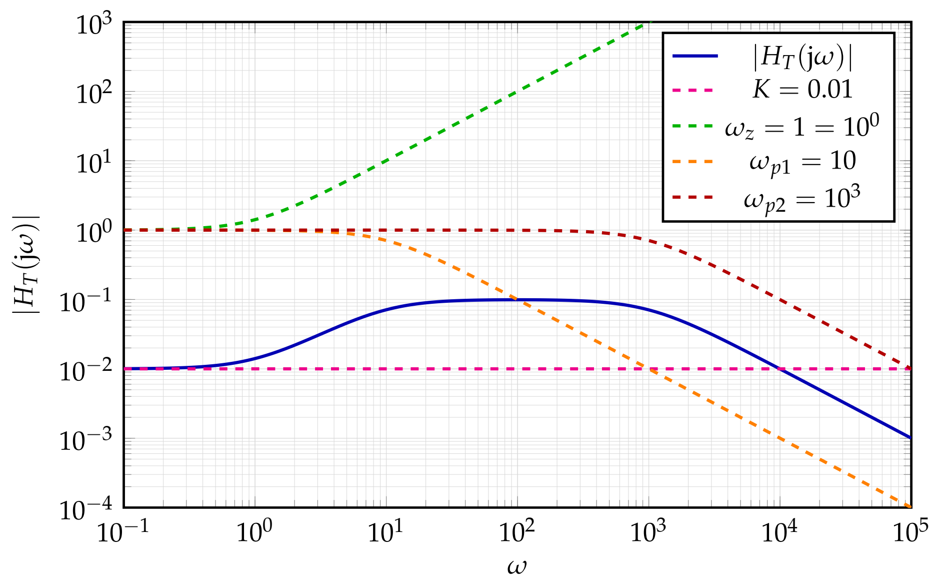

Figure 14: Magnitude Bode Plot of $H_T(\mathrm{j}\omega)$

To provide an analysis for this Bode plot, we see that the plot starts off at $K = 0.01$. Then at $\omega_z = 1$, it starts rising with slope 1. When it hits the pole at $\omega_{p1} = 10$, the slope of 1 is cancelled out by the -1 slope that the pole provides. Then the Bode plot stays constant until $\omega_{p2} = 10^3$ at which it drops off with a slope of 1. We've provided Bode plots of the individual terms to give you a sense of how we "add" Bode plots together.

We can also plot the phase in fig. 15, in a very similar way using our building blocks:

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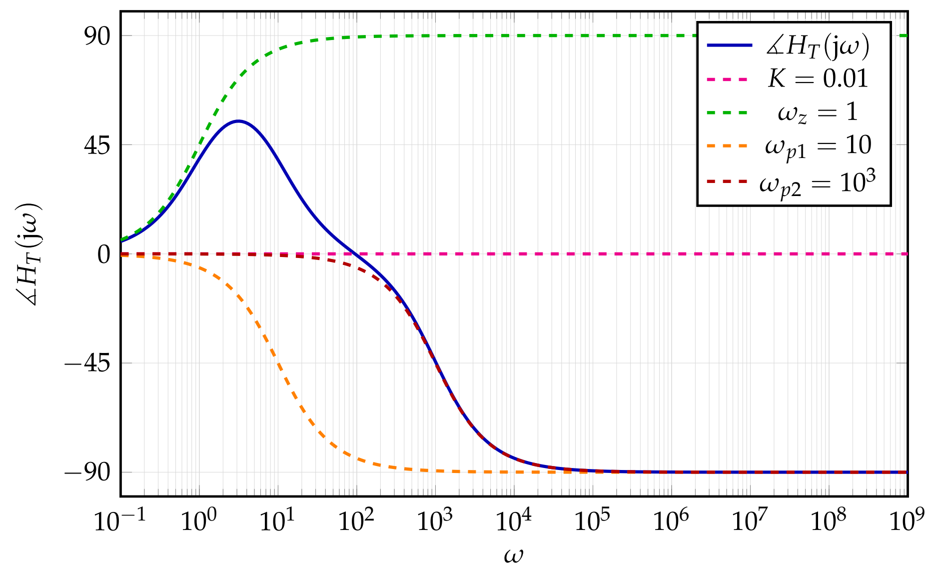

Figure 15: Phase Bode Plot of $H_T(j\omega)$

Zero at the Origin

In our final example, we examine the effects of a zero at the origin. Only the final results are shown; the intermediate building blocks are left fo the reader to consider. We are given the transfer function eq. (15) in rational form.

$$H_O(\omega) = 0.1\frac{(j\omega)(1 + \frac{j\omega}{10^6})}{(1 + \frac{j\omega}{10^2})^2} \tag{15}$$

Our magnitude plot is in fig. 16.

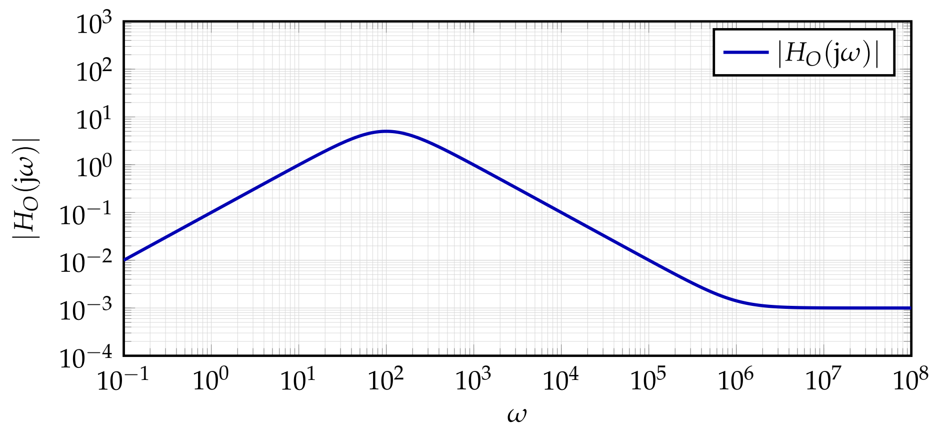

Figure 16: Magnitude Bode Plot of $H_)(j\omega)$

Since there is a zero at the origin, the plot will initially start with a slope of 1. There are no additional zeros or poles before $\omega = 1$, so we can approximate $|H_O(1)| = K = 0.1$. Then the double pole at $\omega_p = 10^2$ provides a slope of -2 that will cancel out the slope of 1 making the overall slope after ω_p equal to -1. Lastly, there is a zero at $\omega_z = 10^6$ and we see that the addition of a slope of 1 makes $|H_O(\omega)|$ remains constant after ω_z.

And for the phase, we have the plot in fig. 17.

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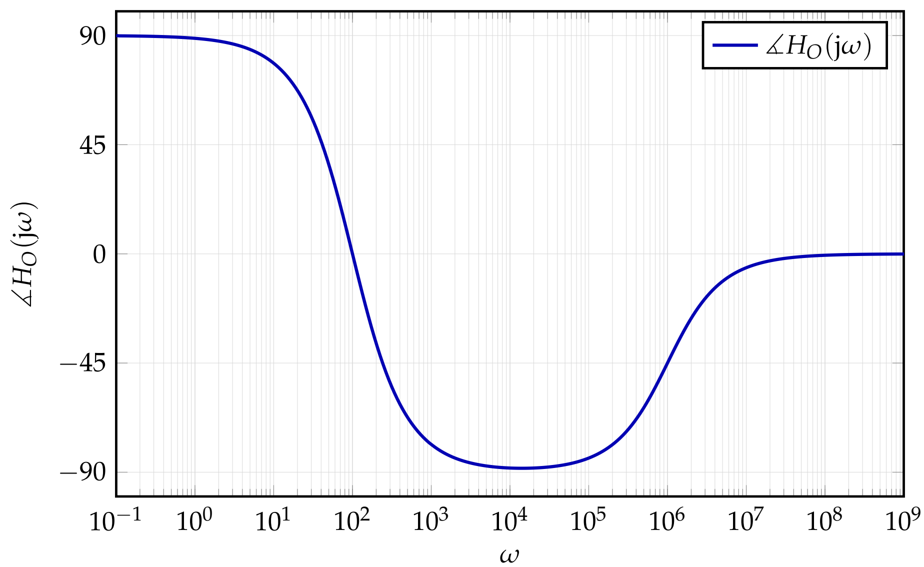

Figure 17: Phase Bode Plot of $H_O(j\omega)$

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