Note 4: Inductors and RLC Circuits

1 Inductors

1.1 Introduction to Inductors

Here, we introduce a new passive component, the inductor. This new component will help us design more interesting circuits and introduce oscillations within our circuits.

Definition 1 (Inductor)
An inductor is denoted as in Figure 1.

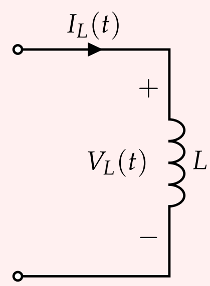

Figure 1: Example Inductor Circuit

The voltage across the inductor is related to its current as follows:

$$V_L(t) = L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} \tag{1}$$

where L is the *inductance* of the inductor. The SI unit of inductance is the Henry (H).
The following are important facts about inductors:

1. The voltage across an inductor cannot change instantaneously.
2. Immediately after a current is passed through the inductor, the inductor acts as an open circuit, but as $t \to \infty$, the inductor acts like a short.

Notice that the voltage-current relationship written in eq. (1) is similar to that of a capacitor, but with voltage and current swapped. The short term and long term behavior of inductors and capacitors are also opposites of each other.

Theorem 2 (Series Equivalence)
Consider the two inductors in series configuration in Figure 2, and suppose we wish to find the series equivalent as in Figure 3.

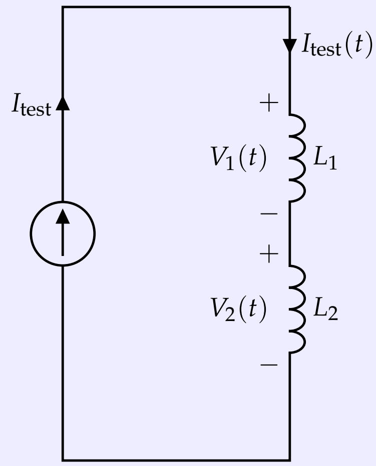

Figure 2: Series Inductor Circuit

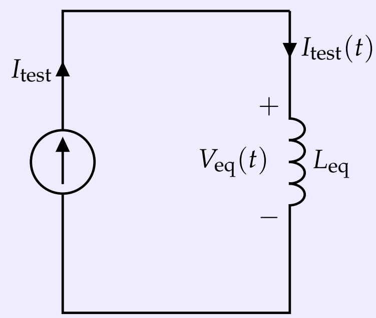

Figure 3: Equivalent Series Inductor Circuit

The equivalent series inductance is $L_{eq} = L_1 + L_2$.

Proof. We use the test current source, $I_{test}(t)$, depicted in Figure 2 and Figure 3 to find the equivalent voltage across both inductors, i.e., $V_{eq}(t)$. Using KVL, we have

$$V_1(t) + V_2(t) = V_{eq}(t) \tag{2}$$

$$L_1 \frac{dI_L(t)}{dt} + L_2 \frac{dI_L(t)}{dt} = V_{eq}(t) \tag{3}$$

$$\underbrace{(L_1 + L_2)}_{L_{eq}} \frac{dI_L(t)}{dt} = V_{eq}(t) \tag{4}$$

as desired. □

Theorem 3 (Parallel Equivalence)
Consider the two inductors in parallel configuration in Figure 4, and suppose we wish to find the parallel equivalent as in Figure 5.

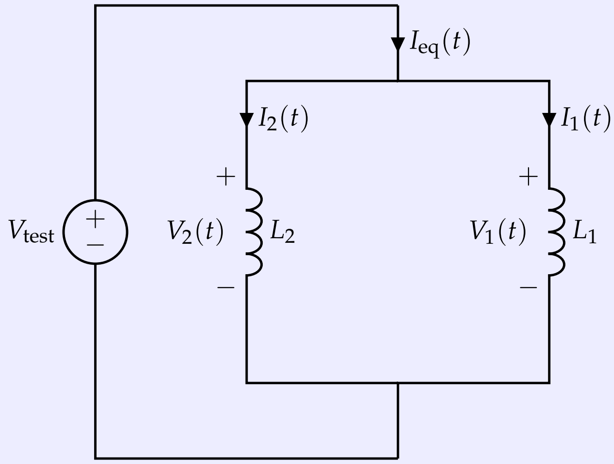

Figure 4: Parallel Inductor Circuit

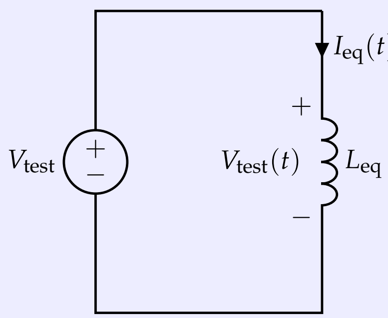

Figure 5: Equivalent Parallel Inductor Circuit

The equivalent inductance is given by $L_{eq} = \left(\frac{1}{L_1} + \frac{1}{L_2}\right)^{-1}$.

Proof. We can apply the test voltage V_{test} as depicted in Figure 4 and Figure 5 to find the equivalent current through both inductors, i.e., $I_{eq}(t)$. By NVA, we have that

$$V_1(t) = V_2(t) = V_{test}(t) \tag{5}$$

$$L_1 \frac{dI_1}{dt} = L_2 \frac{dI_2}{dt} = L_{eq} \frac{dI_{eq}}{dt} \tag{6}$$

and from KCL we have

$$I_{eq}(t) = I_1(t) + I_2(t) \tag{7}$$

$$\frac{dI_{eq}}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt} \tag{8}$$

$$\frac{dI_{eq}}{dt} = \frac{L_{eq}}{L_1} \frac{dI_{eq}}{dt} + \frac{L_{eq}}{L_2} \frac{dI_{eq}}{dt} \tag{9}$$

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} \tag{10}$$

$$L_{eq} = \left(\frac{1}{L_1} + \frac{1}{L_2}\right)^{-1} \tag{11}$$

as desired. □

1.2 OPTIONAL: Physics behind Inductors

Inductors store energy in a magnetic field. In the same way that a capacitor separates charge (Q) and this leads to an electric field ($\vec{E}$), anytime current flows down a conductor, it creates a magnetic field ($\vec{B}$), and this magnetic field can store energy. Inductors' behavior can be described using **Faraday's Law of Induction**.

The magnitude of magnetic field created by a straight wire is pretty small, so we usually use other geometries to create useful inductances. A **solenoid** is a good example, where we wind a wire around a conductor like a copper rod:

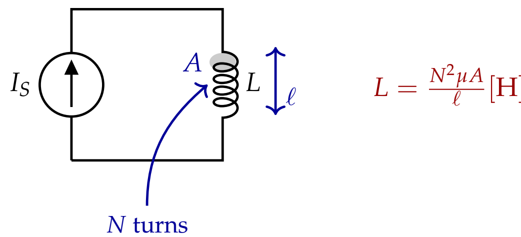

Figure 6: The Inductance of a Solenoid: a wire coiled around something.

Note that the inductance (L) depends on the **geometry** and a material property called **magnetic permeability** (μ) of the solenoid core material. In the case of the solenoid in fig. 6, the inductance depends on the number of turns (N), the length of the solenoid (l) and the area (A) of the loops. Inductors are useful in many applications such as wireless communications, chargers, DC-DC converters, key card locks, transformers in the power grid, etc. But in many high speed applications, their presence might be undesirable as they create delays in the time response of the circuit (analogous to capacitors).

2 LC Tank Example

This section is an extended example to demonstrate how inductors can create oscillations in circuits. Consider the *LC Tank* circuit depicted in Figure 7.

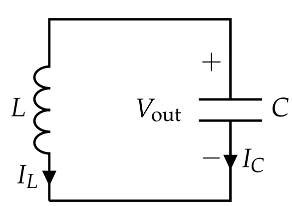

Figure 7: An LC Tank.

We can model $V_{\text{out}}(t)$ using differential equations. Suppose that $V_{\text{out}}(0) = 0$ and $I_L(0) = 1\,\text{A}$.

2.1 Deriving the Differential Equations

We will use KCL and NVA to derive the system of differential equations that models this circuit. NVA gives us

$$V_L = V_C = V_{\text{out}} \tag{12}$$

KCL gives us

$$I_L = -I_C = -C\frac{\mathrm{d}V_{\text{out}}}{\mathrm{d}t} \tag{13}$$

$$\frac{\mathrm{d}V_{\text{out}}}{\mathrm{d}t} = -\frac{1}{C}I_L \tag{14}$$

and NVA again gives us

$$V_L = V_{\text{out}} = L\frac{\mathrm{d}I_L}{\mathrm{d}t} \tag{15}$$

$$L\frac{\mathrm{d}I_L}{\mathrm{d}t} = V_{\text{out}} \tag{16}$$

$$\frac{\mathrm{d}I_L}{\mathrm{d}t} = \frac{1}{L}V_{\text{out}} \tag{17}$$

Notice that we have derivatives of $I_L(t)$ and $V_L(t)$, so we can make these state variables. Arranging this as a matrix differential equation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\begin{bmatrix} V_{\text{out}} \\ I_L \end{bmatrix}}_{\vec{x}(t)} = \underbrace{\begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} V_{\text{out}} \\ I_L \end{bmatrix}}_{\vec{x}(t)} \tag{18}$$

2.2 Solving the Matrix Differential Equation

It happens to be the case A is diagonalizable here. We can solve this matrix differential equation using the technique from the previous note: by first diagonalizing, performing a change of basis, solving a diagonal system, and then undoing the change of basis. We can find the eigenvalues by solving for λ in

$$\det\{A - \lambda I_2\} = 0 \tag{19}$$

which yields $\lambda_1 = \mathrm{j}\frac{1}{\sqrt{LC}}$ and $\lambda_2 = -\mathrm{j}\frac{1}{\sqrt{LC}}$. We can find $\vec{v}_1$, the eigenvector for λ_1, by finding a basis for $\text{Null}(A - \lambda_1 I)$. Computing this gives $\vec{v}_1 = \begin{bmatrix} \mathrm{j}\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}$. We perform a similar operation with λ_2 and obtain $\vec{v}_2 = \begin{bmatrix} -\mathrm{j}\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}$. Hence, we have

$$\Lambda = \begin{bmatrix} \mathrm{j}\frac{1}{\sqrt{LC}} & 0 \\ 0 & -\mathrm{j}\frac{1}{\sqrt{LC}} \end{bmatrix} \tag{20}$$

$$V = \begin{bmatrix} \mathrm{j}\sqrt{\frac{L}{C}} & -\mathrm{j}\sqrt{\frac{L}{C}} \\ 1 & 1 \end{bmatrix} \tag{21}$$

$$\implies V^{-1} = \frac{1}{2\mathrm{j}\sqrt{\frac{L}{C}}}\begin{bmatrix} 1 & \mathrm{j}\sqrt{\frac{L}{C}} \\ -1 & \mathrm{j}\sqrt{\frac{L}{C}} \end{bmatrix} \tag{22}$$

The new differential equation for $\vec{\tilde{x}}(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{x}}(t) = \begin{bmatrix} \mathrm{j}\frac{1}{\sqrt{LC}} & 0 \\ 0 & -\mathrm{j}\frac{1}{\sqrt{LC}} \end{bmatrix} \vec{\tilde{x}}(t) \tag{23}$$

with initial condition $\vec{\tilde{x}}(0) = V^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Solving this diagonal system, we see that

$$\vec{\tilde{x}}(t) = \begin{bmatrix} \frac{1}{2}\mathrm{e}^{\mathrm{j}\frac{t}{\sqrt{LC}}} \\ \frac{1}{2}\mathrm{e}^{-\mathrm{j}\frac{t}{\sqrt{LC}}} \end{bmatrix} \tag{24}$$

Undoing the change of variables to find $\vec{x}(t)$, we obtain

$$\vec{x}(t) = V\vec{\tilde{x}}(t) \tag{25}$$

$$= \begin{bmatrix} \mathrm{j}\sqrt{\frac{L}{C}} & -\mathrm{j}\sqrt{\frac{L}{C}} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\mathrm{e}^{\mathrm{j}\frac{t}{\sqrt{LC}}} \\ \frac{1}{2}\mathrm{e}^{-\mathrm{j}\frac{t}{\sqrt{LC}}} \end{bmatrix} \tag{26}$$

$$= \begin{bmatrix} \sqrt{\frac{L}{C}}\left(\frac{\mathrm{j}}{2}\mathrm{e}^{\mathrm{j}\frac{t}{\sqrt{LC}}} - \frac{\mathrm{j}}{2}\mathrm{e}^{-\mathrm{j}\frac{t}{\sqrt{LC}}}\right) \\ \frac{1}{2}\mathrm{e}^{\mathrm{j}\frac{t}{\sqrt{LC}}} + \frac{1}{2}\mathrm{e}^{-\mathrm{j}\frac{t}{\sqrt{LC}}} \end{bmatrix} \tag{27}$$

Using Euler's formula ($\mathrm{e}^{\mathrm{j}\theta} = \cos(\theta) + \mathrm{j}\sin(\theta)$), we can simplify the above to obtain

$$\vec{x}(t) = \begin{bmatrix} -\sqrt{\frac{L}{C}}\sin\left(\frac{t}{\sqrt{LC}}\right) \\ \cos\left(\frac{t}{\sqrt{LC}}\right) \end{bmatrix} \tag{28}$$

so we have $V_{\text{out}}(t) = -\sqrt{\frac{L}{C}}\sin\left(\frac{t}{\sqrt{LC}}\right)$ and $I_L(t) = \cos\left(\frac{t}{\sqrt{LC}}\right)$.

2.3 Visualizing $V_{\text{out}}(t)$, $I_L(t)$, and Energy

A plot of $I_L(t)$ and $V_{\text{out}}(t)$ will resemble the graph in Figure 8.

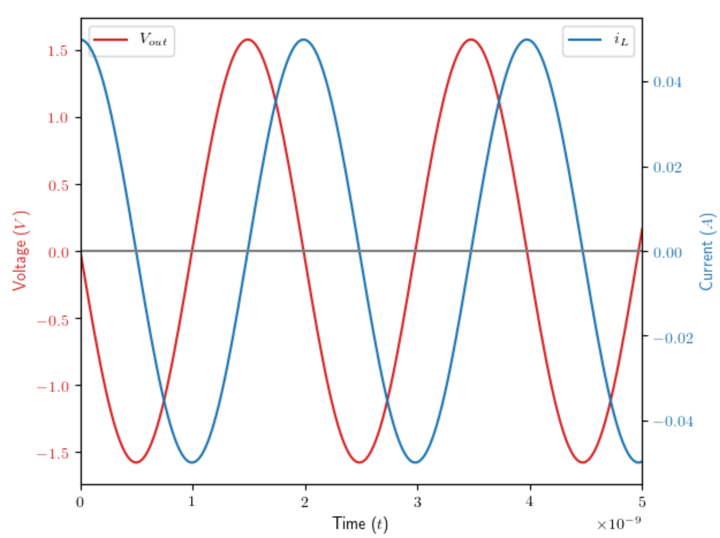

Figure 8: Voltage and Current response of LC Tank

We can find the energy in the capacitor and inductor respectively:

$$E_C = \frac{1}{2}CV_{\text{out}}^2 = \frac{L}{2}\sin^2\left(\frac{t}{\sqrt{LC}}\right) \tag{29}$$

$$E_L = \frac{1}{2}LI_L^2 = \frac{L}{2}\cos^2\left(\frac{t}{\sqrt{LC}}\right) \tag{30}$$

Notice that $E_C + E_L = \frac{L}{2}$, so the energy is constant over time. This is expected, since physics tells us that energy in this closed system should be conserved. A plot of E_C and E_L will resemble Figure 9.

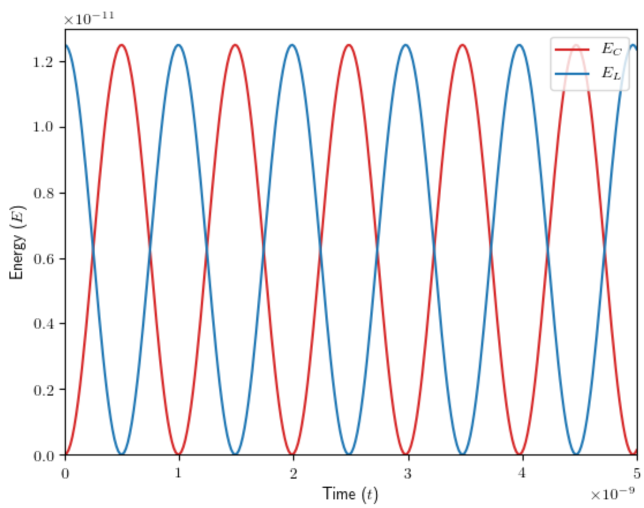

Figure 9: Energy stored in Inductor and Capacitor. Notice the sum is constant.

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