Note 7: Transfer Function Plots

Overview

Having analyzed our first order filters and gone through a design example in the previous Note to show why filter design is important, we will now plot their transfer functions $H(\mathrm{j}\omega)$ (or frequency responses). In the previous Note, we generated tables of $|H(\mathrm{j}\omega)|$, $\measuredangle H(\mathrm{j}\omega)$ at certain key values of ω_c, and while this gave some intuition, it didn't really show what happens at intermediate frequencies. There is immense value in visualizing transfer functions across a wide range of frequencies.

In the next note, we will introduce the concept of piecewise-linear numerical approximations to these plots, called Bode Plots. These will not only prove useful in plotting a filter's frequency response by hand (as opposed to with a computer or plotting software), but will also help us better understand filter behavior.

1 Transfer Function Plots

When we plot transfer functions, we plot the frequency and magnitude on *logarithmic scales* (called log-log plots), and the phase angle is plotted on a linear scale, in either degrees or radians. We use the logarithmic scale because it allows us to break up complex transfer functions into its constituent components, which makes plotting by hand much easier. Let's start by generating transfer function plots for low-pass and high-pass first-order filters, which will build our intuition. We will soon see how the analysis of more complex transfer functions can be broken down into parts.

1.1 Low-pass Filter

Recall our generalized model of a low-pass filter (perhaps RC or LR) as described in the previous note:

$$H_{\mathrm{LP}}(\mathrm{j}\omega) = \frac{1}{1 + \mathrm{j}\omega/\omega_c} \tag{1}$$

We plot the magnitude of the frequency response in fig. 1 assuming $\omega_c = 10^6$ (on log-log scales).

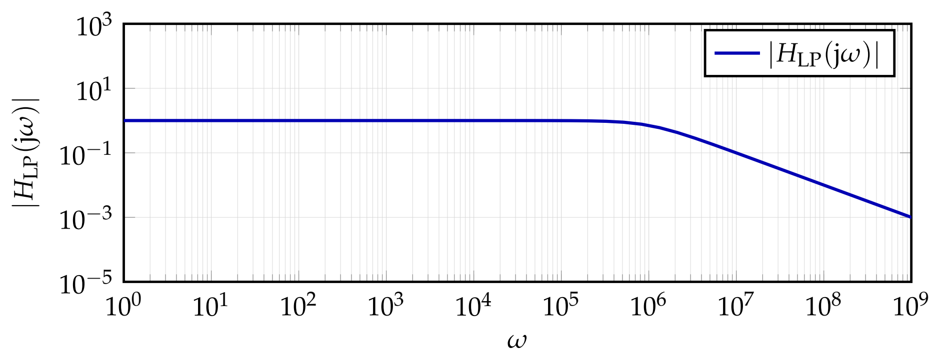


Figure 1: RC low-pass magnitude plot.

Note how $|H_{\mathrm{LP}}(\mathrm{j}\omega)|$ is very close to 1 for $\omega < \omega_c$ and $|H_{\mathrm{LP}}(\mathrm{j}\omega)|$ starts dropping off with slope -1 after ω_c. We can observe 3 distinct regions on the plot, which are defined in terms of an important quantity

— the circuit's cutoff frequency. See the previous note for a recap of the cutoff frequency definition and significance.

- $\omega \ll \omega_c$: $\implies$ $\mathrm{j}\omega/\omega_c \approx 0$. So, $H_{\mathrm{LP}}(\mathrm{j}\omega) \approx 1$ and $|H_{\mathrm{LP}}(\mathrm{j}\omega)| \approx 1$.
- $\omega = \omega_c$: $\implies$ $H(\mathrm{j}\omega_c) = \frac{1}{1+\mathrm{j}}$. So, $|H_{\mathrm{LP}}(\mathrm{j}\omega_c)| = \frac{1}{\sqrt{2}}$.[1]
- $\omega \gg \omega_c$: $\implies$ $\omega/\omega_c \gg 1$. Therefore $H_{\mathrm{LP}}(\mathrm{j}\omega) \approx -\mathrm{j}\frac{\omega_c}{\omega}$. So, $|H_{\mathrm{LP}}(\mathrm{j}\omega)| \approx \frac{\omega_c}{\omega}$. On a log scale, this means that $\log|H_{\mathrm{LP}}(\mathrm{j}\omega)| \approx \log\omega_c - \log\omega$ explaining behavior of dropping off with slope -1.[2]

Now, let's plot the phase of $H_{\mathrm{LP}}(\mathrm{j}\omega)$ in fig. 2 (on a log-linear scale).

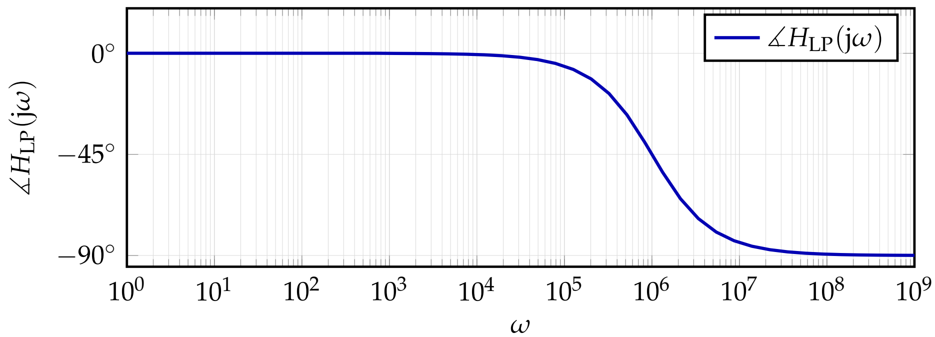


Figure 2: RC low-pass phase plot.

$\measuredangle H_{\mathrm{LP}}(\mathrm{j}\omega)$ is very close to 0 for $\omega < 0.1\omega_c$ and $\measuredangle H_{\mathrm{LP}}(\mathrm{j}\omega)$ is approximately $-\frac{\pi}{2}$ for $\omega > 10\omega_c$. The reason why we examine the specific regions below will become clearer in the next Note, on linear approximations.

- $\omega \ll 0.1\omega_c \implies \mathrm{j}\omega/\omega_c \approx 0$. So, $H_{\mathrm{LP}}(\mathrm{j}\omega) \approx 1$ and $\measuredangle H_{\mathrm{LP}}(\mathrm{j}\omega) \approx 0$.
- $\omega = 0.1\omega_c \implies H_{LP}(0.1\mathrm{j}\omega_c) = \frac{1}{1+\mathrm{j}\,0.1}$ and $\measuredangle H_{\mathrm{LP}}(0.1\mathrm{j}\omega_c) \approx -6°$.[3]
- $\omega = \omega_c \implies H_{LP}(\mathrm{j}\omega_c) = \frac{1}{1+\mathrm{j}}$ and $\measuredangle H_{\mathrm{LP}}(\mathrm{j}\omega_c) = -45°$.
- $\omega = 10\omega_c \implies H_{LP}(10\mathrm{j}\omega_c) = \frac{1}{1+\mathrm{j}\,10}$ and $\measuredangle H_{\mathrm{LP}}(10\mathrm{j}\omega_c) \approx -84°$.
- $\omega \gg 10\omega_c \implies \omega/\omega_c \gg 10$. So, $H_{\mathrm{LP}}(\mathrm{j}\omega) \approx -\mathrm{j}\cdot 0$ [4] and $\measuredangle H_{\mathrm{LP}}(\mathrm{j}\omega) \approx -90°$.

We can now better understand the values of the magnitude and phase at $0.1\omega_c$, ω_c, $10\omega_c$ (as seen in the tables of the previous note).

1.2 High-pass Filter

We can similarly analyze our generalized high-pass filter model (CR, RL):

$$H_{\mathrm{HP}}(\mathrm{j}\omega) = \frac{\mathrm{j}\omega/\omega_c}{1+\mathrm{j}\omega/\omega_c} \tag{2}$$

Plotting the magnitude of the frequency response, again assuming $\omega_c = 10^6$, yields fig. 3.

[1] Hopefully, after some exposure to complex numbers, it is more clear now how $|1+\mathrm{j}| = \sqrt{1^2+1^2} = \sqrt{2}$.

[2] Recall that the line $y = mx + b$ has slope m. In this case $y = \log|H_{\mathrm{LP}}(\mathrm{j}\omega)|$ and $x = \log|\omega|$.

[3] This value comes from how $\measuredangle\frac{z_1}{z_2} = \measuredangle z_1 - \measuredangle z_2$, and $\measuredangle a + jb = \mathrm{atan2}(b, a)$. So here, we have $\mathrm{atan2}(0,1) - \mathrm{atan2}(0.1,1) = -5.71° \approx -6°$

[4] The magnitude will always be slightly greater than 0, meaning its phase will still be very close to $-90°$

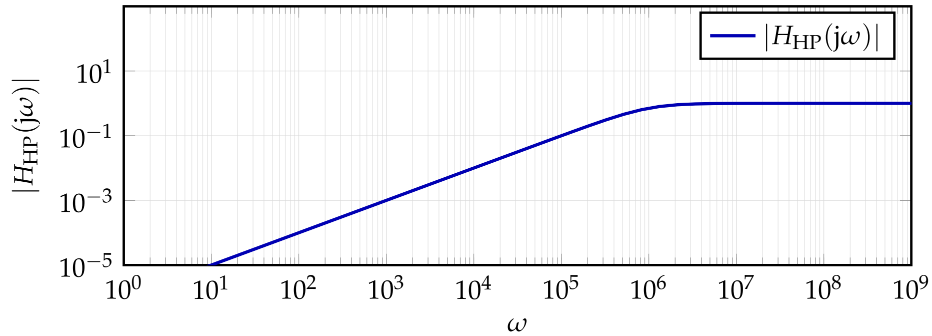


Figure 3: RC high-pass magnitude plot.

Here, $|H_{\mathrm{HP}}(\mathrm{j}\omega)|$ rises with slope 1 for $\omega < \omega_c$ and $|H_{\mathrm{HP}}(\mathrm{j}\omega)| \approx 1$ after ω_c. We analyze the plot in the same 3 regimes as before (smaller than, equal to, and larger than ω_c):

- $\omega \ll \omega_c$, then $\omega/\omega_c \ll 1$. Therefore $H_{\mathrm{HP}}(\mathrm{j}\omega) \approx \mathrm{j}\frac{\omega}{\omega_c}$ which implies $|H_{\mathrm{HP}}(\mathrm{j}\omega)| \approx \frac{\omega}{\omega_c}$. On a log scale, this means that $\log|H_{\mathrm{HP}}(\mathrm{j}\omega)| \approx \log\omega - \log\omega_c$, which explains the rising slope of 1.
- $\omega = \omega_c$, then $H(\omega_c) = \frac{\mathrm{j}}{1+\mathrm{j}}$ meaning $|H_{\mathrm{HP}}(\mathrm{j}\omega_c)| = \frac{1}{\sqrt{2}}$
- $\omega \gg \omega_c$, then $\omega/\omega_c \gg 1$. Therefore $H_{\mathrm{HP}}(\mathrm{j}\omega) \approx 1$ which implies $|H_{\mathrm{HP}}(\mathrm{j}\omega)| \approx 1$.

Now let's plot the phase of the transfer function $H_{\mathrm{HP}}(\mathrm{j}\omega)$.

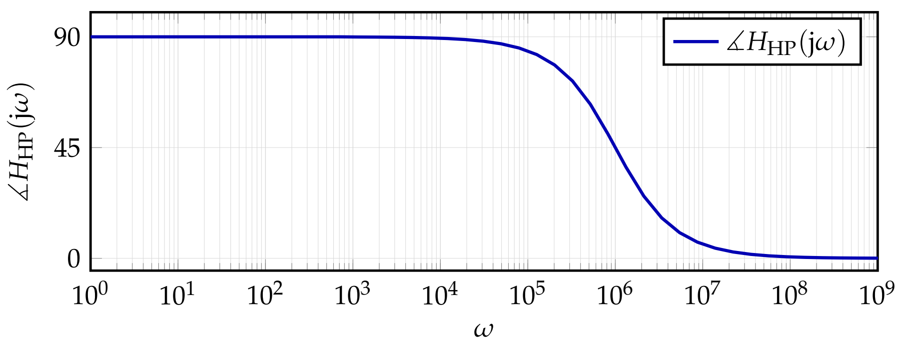


Figure 4: RC high-pass phase plot.

$\measuredangle H_{\mathrm{HP}}(\mathrm{j}\omega)$ is very close to $\frac{\pi}{2}$ for $\omega < 0.1\omega_c$ and $\measuredangle H_{\mathrm{HP}}(\mathrm{j}\omega)$ is approximately 0 for $\omega > 10\omega_c$.

- $\omega \ll 0.1\omega_c \implies \mathrm{j}\omega/\omega_c \approx 0$. So, $H_{\mathrm{HP}}(\mathrm{j}\omega) \approx 0$ (but slightly positive) and so $\measuredangle H_{\mathrm{HP}}(\mathrm{j}\omega) \approx 90°$.
- $\omega = 0.1\omega_c \implies H_{HP}(0.1\mathrm{j}\omega_c) = \frac{\mathrm{j}\,0.1}{1+\mathrm{j}\,0.1}$ and $\measuredangle H_{\mathrm{HP}}(0.1\mathrm{j}\omega_c) \approx 84°$.
- $\omega = \omega_c \implies H_{HP}(\mathrm{j}\omega_c) = \frac{\mathrm{j}}{1+\mathrm{j}}$ and $\measuredangle H_{\mathrm{HP}}(\mathrm{j}\omega_c) = 45°$.
- $\omega = 10\omega_c \implies H_{HP}(10\mathrm{j}\omega_c) = \frac{\mathrm{j}\,10}{1+\mathrm{j}\,10}$ and $\measuredangle H_{\mathrm{HP}}(10\mathrm{j}\omega_c) \approx 6°$.
- $\omega \gg 10\omega_c \implies \omega/\omega_c \gg 10$. So, $H_{\mathrm{HP}}(\mathrm{j}\omega) \approx 1$ and $\measuredangle H_{\mathrm{HP}}(\mathrm{j}\omega) \approx 0°$.

2 Second Order Filters (Cascading)

We will now consider more complex systems and, in doing so, see the value in appropriate visualizations for transfer function behavior.

2.1 Band-Pass Filters

With the knowledge of low-pass filters that block out higher frequencies and high-pass filters that block out lower frequencies, how could we build a filter that lets a specific range of frequncies through? One idea could be to take the output of the low-pass filter and treat it as an input to the high-pass filter.

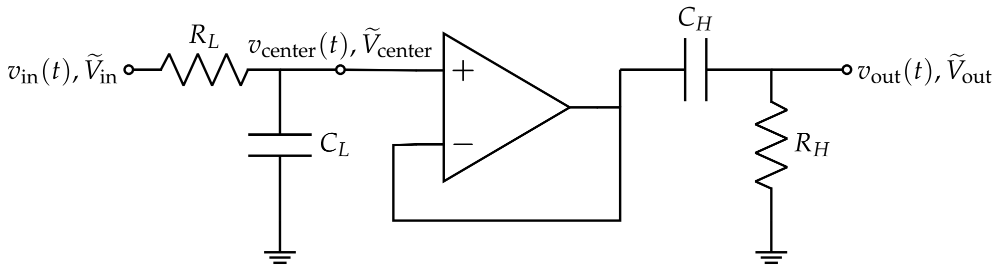


Figure 5: A Buffered Band-Pass filter, composed of low-pass and high-pass filter components.

Notice the op-amp serving as a unity gain buffer between the two filters. It is introduced to prevent the second circuit from loading the first.

The input voltage phasor is $\widetilde{V}_{\text{in}}$ at some frequency ω. Suppose the two filters have cutoff frequencies ω_{LP} and ω_{HP}, respectively. Thus, we can write that:

$$\widetilde{V}_{\text{center}} = H_{\text{LP}}(\mathrm{j}\omega)\widetilde{V}_{\text{in}} \tag{3}$$

Since $v_{\text{center}}(t)$ is the second filter's input, the output voltage phasor $\widetilde{V}_{\text{out}}$ is:

$$\widetilde{V}_{\text{out}} = H_{\text{HP}}(\mathrm{j}\omega)\widetilde{V}_{\text{center}} = H_{\text{HP}}(\mathrm{j}\omega)H_{\text{LP}}(\mathrm{j}\omega)\widetilde{V}_{\text{in}} \tag{4}$$

Thus, the net transfer function $H_{\text{BP}}(\mathrm{j}\omega)$ is:

$$H_{\text{BP}}(\mathrm{j}\omega) = H_{\text{LP}}(\mathrm{j}\omega)H_{\text{HP}}(\mathrm{j}\omega) \tag{5}$$

More generally, placing filters in series produces a circuit whose transfer function is the product of the individual transfer functions. Similarly, for $\measuredangle H_{\text{BP}}(\mathrm{j}\omega)$, we can again plot the sum, $\measuredangle H_{\text{LP}}(\mathrm{j}\omega) + \measuredangle H_{\text{HP}}(\mathrm{j}\omega)$.

We can compute $H_{\text{BP}}(\mathrm{j}\omega)$ symbolically as:

$$H_{\text{BP}}(\mathrm{j}\omega) = H_{\text{LP}}(\mathrm{j}\omega)H_{\text{HP}}(\mathrm{j}\omega) = \frac{1}{1+\mathrm{j}\omega R_L C_L} \cdot \frac{\mathrm{j}\omega R_H C_H}{1+\mathrm{j}\omega R_H C_H} \tag{6}$$

To find the cutoff frequencies of this filter, we can look at the points at which $H_{\text{BP}}(\omega_c) = \frac{1}{\sqrt{2}}$. But based on our approximations from before and from the cuttof frequencies $\omega_{\text{LP}} = \frac{1}{R_L C_L}$ and $\omega_{\text{HP}} = \frac{1}{R_H C_H}$, we can approximate $|H_{\text{LP}}(\mathrm{j}\omega)| \approx \frac{1}{\sqrt{2}} \cdot 1$ and $|H_{\text{HP}}(\mathrm{j}\omega)| \approx 1 \cdot \frac{1}{\sqrt{2}}$. This approximation holds best when the cuttof frequencies are spaced apart.

We've shown a convenient result! The cutoffs for the band-pass filter are identical to the individual cutoffs for the low and high-pass filters. We now plot $H_{\text{BP}}(\mathrm{j}\omega)$ with $\omega_{\text{LP}} = 10^6$ and $\omega_{\text{HP}} = 10^4$ to demonstrate the band-pass behavior. This is the approximate shape of the band-pass filter we constructed in the previous note, with a high-pass and low-pass filter.

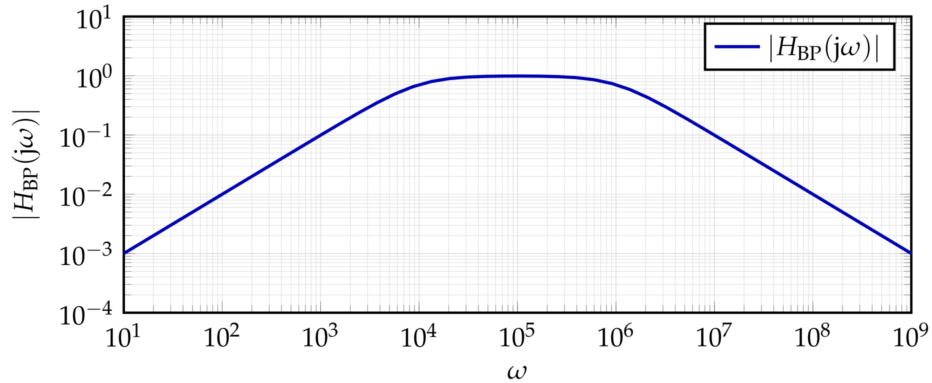


Figure 6: BP magnitude plot.

2.2 Low-Pass Filters

From our analysis of low-pass filters, we saw that $|H(\text{j}\omega)|$ dropped off by a factor of 10 for each factor-of-10 increase in frequency after ω_c. This is a desirable effect, but ideally, we would like to build a filter that drops off at a quicker rate after ω_c. Therefore, let's try cascading two low-pass filters of identical cutoff with a buffer in between. The diagram is exactly like our band-pass filter from before, but with 2 low-pass filters instead of a high-pass and a low-pass.

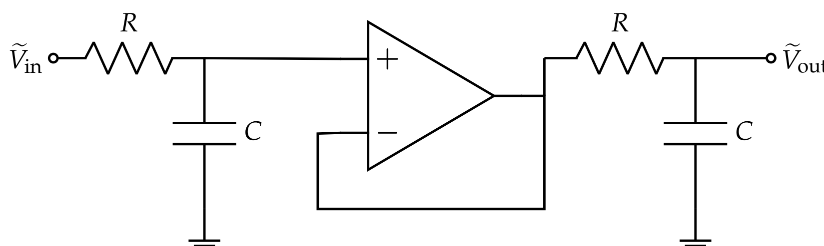


Figure 7: A Buffered Second Order Low-Pass filter. To achieve fast roll-off after ω_c, we must use R, C values that generate the same cutoff in each stage (same RC product). Here, we achieved this condition by setting R and C to be the same values in both filters.

With similar analysis as for the band-pass filter:

$$H_{\text{LP}}(\text{j}\omega) = \frac{1}{(1 + \text{j}\omega RC)^2} \tag{7}$$

We can see that it does indeed drop off at a quicker rate (with slope 2 after the cutoff ω_c.).

3 Higher Order Transfer Functions Plots

The previous section talked about second order filters (with 2 filter stages), but we can generalize similar logic to figure out what happens when we cascade an arbitrary number of stages.

3.1 Cascading More Low-Pass Filters: Example

For example, let's generalize the second order low-pass filter to see what happens when we add more stages with the same cutoff frequency. Why would we want to do this? Well, in the example plot of fig. 8, suppose that we had noise at the input we wanted to reject at $\omega_{\text{noise}} = 1 \times 10^7 \, \frac{\text{rad}}{\text{s}}$. Using a single stage gives us attenuation by a factor of 10 (magnitude of the first-order low-pass transfer function with cutoff $\omega_c = 1 \times 10^6 \, \frac{\text{rad}}{\text{s}}$ is $\frac{1}{10}$ at ω_{noise}). 10% of the noise is still a decent portion, we may want better rejection.

The plots in fig. 8 show what happens to the magnitude of the overall transfer function as we add more than 1 stage.

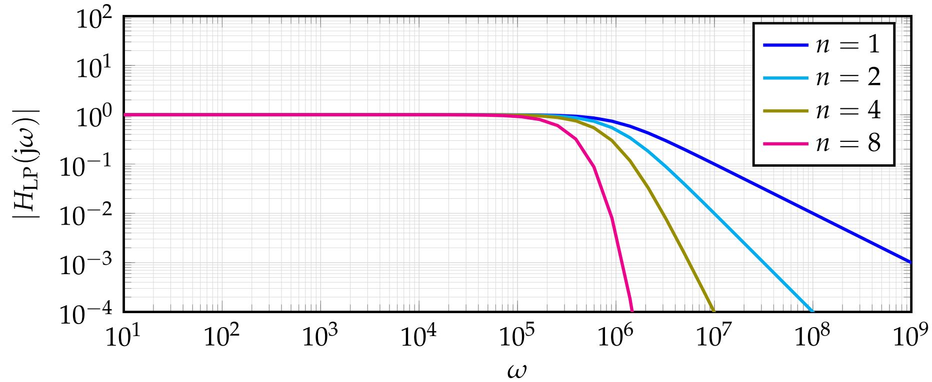


Figure 8: Plots of cascaded LP filters (varying numbers).

From fig. 8, we see that the "actual" cutoff frequency seems to be shifting as a result of the accumulation of stages. Specifically, if we define the cutoff frequency to be the point where we have a $\frac{1}{\sqrt{2}}$ (or -3 dB)drop-off, this will shift to the left as we add more stages. However, we won't worry about this for now; we could compensate for this in various ways using specific filter variations, which aren't in scope here. The main important aspect of the overall circuit to check is how much the cascading attenuates the desired input signal, which will vary depending on the signal frequency. For certain designs, we may need to compromise between attenuating noise and retaining signal.

3.2 Generalized Composition of Buffered Filter Stages

If we have some number n of transfer functions $H_1(\text{j}\omega), H_2(\text{j}\omega), H_3(\text{j}\omega), ..., H_n(\text{j}\omega)$ and we define the overall transfer function $H(\text{j}\omega)$ as the product of them all ($H(\text{j}\omega) = H_1(\text{j}\omega) \cdot H_2(\text{j}\omega) \cdot ... \cdot H_n(\text{j}\omega)$)

$$
\begin{aligned}
|H(\text{j}\omega)| &= |H_1(\text{j}\omega) \cdot H_2(\text{j}\omega) \cdot \; ... \; \cdot H_n(\text{j}\omega)| && (8)\\
&= |H_1(\text{j}\omega)| \cdot |H_2(\text{j}\omega)| \cdot \; ... \; \cdot |H_n(\text{j}\omega)| && (9)\\
&= \prod_{i=1}^{i=n} |H_i(\text{j}\omega)| && (10)\\
\measuredangle H(\text{j}\omega) &= \measuredangle(H_1(\text{j}\omega) \cdot H_2(\text{j}\omega)) \cdot \; ... \; \cdot H_n(\text{j}\omega)) && (11)\\
&= \measuredangle H_1(\text{j}\omega) + \measuredangle H_2(\text{j}\omega) + \; ... \; + \measuredangle H_n(\text{j}\omega) && (12)\\
&= \sum_{i=1}^{i=n} \measuredangle H_i(\text{j}\omega) && (13)
\end{aligned}
$$

In the next note, we will see how to convert this *multiplication* of transfer functions in a log scale into an *addition* of transfer functions in a linear scale (as a result of the properties of logarithms), and this will make hand-composition of transfer function plots easier.

All of the methods we have used thus far to combine and plot transfer functions require us to use a computer, or some kind of plotting software, to get a reasonsably accurate result. In the next note, we will develop a useful technique for plotting such transfer functions by hand, called the Bode Plot (or piecewise-linear) approximation. A lot of that content may feel like review from this note. This process is of great use in performing filter design for various applications, as it enables concrete understanding of how to take several simple filter stages and combine them together to achieve some desired goal (much as we did in composing the band-pass filter in this note).

4 Further Design Considerations and Examples

In this section, we will outline some key facts to keep in mind about selecting filter stages, and their corresponding cutoff frequencies. Some of this content may have been covered in previous notes, but hopefully this serves as a concrete reference. It is recommended to read the Design Example section in the previous note before proceeding.

The general practical application for which we want to use analog filters is when our input signal has sources of noise which are at different frequencies from the signal of interest.[5] In this case, we can apply a cascade of filters, where the cutoff frequencies are selected to keep as much of the desired signal as possible while rejecting (attenuating the magnitude of) the undesired signals as much as possible.

The simplest scenarios involve two frequencies, where one is desired (say ω_{sig}) and the other is noise (ω_{noise}). If $\omega_{\text{sig}} > \omega_{\text{noise}}$, then we use a high-pass filter. Conversely, for $\omega_{\text{sig}} < \omega_{\text{noise}}$, we choose a low-pass filter.

Suppose that for some situation, we have $\omega_{\text{sig}} = 1 \times 10^3 \,\frac{\text{rad}}{\text{s}}$ and $\omega_{\text{noise}} = 40 \times 10^3 \,\frac{\text{rad}}{\text{s}}$, and so we know to use a low-pass filter. How do we choose the cutoff though? There is *always* a tradeoff between rejecting the noise and keeping the signal, so different applications have to be considered individually. For example, suppose your analog-to-digital converter requires a minimum signal amplitude of 10 mV after analog-filtering out noise. If your signal has an input amplitude of 1 mV and you have an op-amp to apply a gain of 12 afterwards, then you will meet the 10 mV threshold.

However, this also means that your analog filter cannot attenuate the signal too much; if the amplitude of the filtered signal is only 0.707 mV (which will happen if your filter has a cutoff frequency directly on the signal frequency with $\omega_{\text{sig}} = \omega_c$), then after applying a gain of 12, your signal has an amplitude of $\approx 8.5\,\text{mV} < 10\,\text{mV}$. As a result, we can solve for the *minimum* low-pass cutoff frequency that will still keep our signal above the required magnitude of $\frac{10}{12}$mV (so it's exactly 10 mV after being gained up by the op-amp.)

$$\left|H(\mathrm{j}\omega_{\text{sig}})\right| = 0.83 \tag{14}$$

$$= \frac{|1|}{\left|1 + \mathrm{j}\frac{\omega_{\text{sig}}}{\omega_c}\right|} \tag{15}$$

[5] At the moment, if sources of noise have the same frequency as our, our current filtering techniques will find it challenging to handle. We will discuss situations in which filtering can actually help reject the undesired frequencies.

$$= \frac{1}{\sqrt{1^2 + \frac{\omega_{\text{sig}}^2}{\omega_c^2}}} \tag{16}$$

$$\implies \frac{\omega_{\text{sig}}}{\omega_c} = 0.67 \tag{17}$$

$$\implies \frac{\omega_c}{\omega_{\text{sig}}} = 1.49 \tag{18}$$

What does this mean? Well, we solved for how large the cutoff frequency ω_c had to be in comparison to ω_{sig} to ensure the signal stayed above a certain magnitude, and we now have a concrete ratio based on our ADC's physical constraints. That is, if $\omega_{\text{sig}} = 1 \times 10^3 \, \frac{\text{rad}}{\text{s}}$ and $\omega_{\text{noise}} = 40 \times 10^3 \, \frac{\text{rad}}{\text{s}}$, and we set our low-pass cutoff frequency to be $\omega_c = 1.49 \times 10^3 \, \frac{\text{rad}}{\text{s}}$, then our ω_{sig} amplitude at the output will be exactly 10 mV as desired. What happens to the magnitude of the noise? We can compute this explicitly as $|H(\mathrm{j}\omega_{\text{noise}})| = \frac{|1|}{\left|1+\mathrm{j}\frac{\omega_{\text{noise}}}{\omega_c}\right|} = 0.0359$. This means we've only passed through about 3.6% of the noise, while keeping the requisite magnitude of signal. Not bad!

Now, we can consider motivate the need for higher-order filters by considering a variation of this problem, where ω_{noise} is much closer to ω_{sig} (say, $6 \times 10^3 \, \frac{\text{rad}}{\text{s}}$). Now, we need a more aggressive rejection of the noise, because our previous scheme would keep nearly $\frac{|1|}{\left|1+\mathrm{j}\frac{\omega_{\text{noise}}}{\omega_c}\right|} = 24.1\%$ of the noise, which might be too much if the noise magnitude is about equal to signal magnitude. The way to achieve this more aggressive rejection is by cascading multiple low-pass filters with the same cutoff, each separated by a unity-gain buffer. The overall transfer function will become $|H(\mathrm{j}\omega)| = \frac{1}{\left(1+\mathrm{j}\frac{\omega}{\omega_c}\right)^n}$, and we can choose the smallest n[6] that suffices to achieve sufficient rejection of the noise while also maintaining the signal amplitude. This may also require tuning the cutoff frequency of the base low-pass filter to make sure we meet all the requirements, or we may need a stronger op-amp with higher gain after the filter, to allow for more signal attenuation.

4.1 Cutoff Frequency Selection: Rule of Thumb

The above analysis makes sense in tightly-constrained scenarios, but what's a good rule of thumb to start with? We step back and think about the goal. With something like a low-pass filter, we want to effectively keep as much of the signal as possible (have ω_c be far away from ω_{sig}) while also rejecting the noise as much as possible (have ω_c be far away from ω_{noise}). Clearly, these are competing effects, because our analysis and intuition from before tell us that $\omega_{\text{sig}} \leq \omega_c \leq \omega_{\text{noise}}$ for situations where a low-pass filter makes sense.

Now, the most balanced approach to take is to make ω_c be the same distance from the signal and noise frequencies, but *this distance must be computed in the log-scale*. Recall that when we discuss concepts like slope, we are speaking about the log-log magnitude plot. So, what's the analogous quantity to the arithmetic midpoint on a log scale? It's the *geometric mean*. That is, we may choose the following cutoff frequency as a rule of thumb to start design:

$$\omega_c = \sqrt{\omega_{\text{sig}} \, \omega_{\text{noise}}} \tag{19}$$

Now, all of the methods we have used thus far to combine and plot transfer functions require us to use a computer, or some kind of plotting software, to get a reasonably accurate result. In the next note, we will develop a useful technique for plotting such transfer functions by hand, called the Bode Plot (or piecewise-linear) approximation. A lot of that content may feel like review from this note. This process is of great

[6] Remember, hardware costs money! We shouldn't use more stages than needed, when possible.

use in performing filter design for various applications, as it enables concrete understanding of how to take several simple filter stages and combine them together to achieve some desired goal (much as we did in composing the band-pass filter in this note).

4.2 Decibels Scale

We define the decibel as the following:

$$20\log_{10}(|H(\mathrm{j}\omega)|) = |H(\mathrm{j}\omega)| \text{ [dB]} \tag{20}$$

The origin of the decibel comes from looking at the ratio of the output and input power of the system. It is also partially a value purely arising from convention, and the days of using slide rules as computation aids.

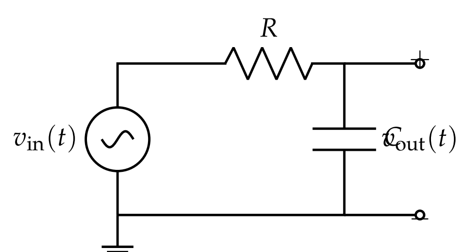
$$|H(\mathrm{j}\omega)| \text{ [dB]} = 10\log\left|\frac{P_{\text{out}}}{P_{\text{in}}}\right| = 10\log\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right|^2 = 20\log\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right| \tag{21}$$

This means that 20 dB per decade is equivalent to one order of magnitude. This scale is particularly useful when performing multiplication of magnitudes as additions in the decibel scale, as seen in discussion and homework.

Appendix A Time Constant

When computing the cutoff frequency for a first order low-pass filter, we noticed that the $\omega_c = \frac{1}{RC} = \frac{1}{\tau}$. Here, we draw the connection between time constants and cutoff frequencies.

Recall from the note on differential equations that we defined the time constant of a first-order circuit to be the point at which the response $v_C(t)$ to a constant input was $1 - \mathrm{e}^{-1}$ away from its steady state value. With this in mind, let's try plugging in an exponential input $v_{\text{in}}(t) = V_0\mathrm{e}^{\mathrm{j}\omega t}$ into an RC circuit and see what happens.[7]



The differential equation for this circuit is

$$\frac{\mathrm{d}}{\mathrm{d}t}v_{\text{out}}(t) = \lambda\left(v_{\text{out}}(t) - V_0\mathrm{e}^{\mathrm{j}\omega t}\right) \tag{22}$$

for $\lambda = -\frac{1}{\tau}$. In Note 3 we showed that the steady state value of this differential equation is

$$v_{ss}(t) = \frac{-\lambda}{\mathrm{j}\omega - \lambda}V_0\mathrm{e}^{\mathrm{j}\omega t} \tag{23}$$

[7] We should be inputting $v_{\text{in}}(t) = V_0\cos(\omega t)$ but we choose $\mathrm{e}^{\mathrm{j}\omega t}$ since it provides the same result while simplifying the math.

Therefore, plugging in for $\lambda = -\frac{1}{\tau}$, it follows that

$$v_{ss}(t) = \frac{1}{1 + \mathrm{j}\omega\tau} V_0 \mathrm{e}^{\mathrm{j}\omega t} \tag{24}$$

Notice that $H(\omega) = \frac{1}{1+\mathrm{j}\omega\tau}$ and the cutoff arises naturally as $\omega_c = \frac{1}{\tau}$. We can also realize that at steady state, $H(\omega)$ is in fact the eigenvalue for the differential equation with eigenfunction $\mathrm{e}^{\mathrm{j}\omega t}$. This is a crucial connection between differential equations and the frequency response of a linear system that you will see in later half of the course and in courses like EE120.

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