Note 18: Linearization

1 Overview

Thus far, we have covered only *linear* control models, say a difference equation of the form

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] \tag{1}$$
$$\vec{x}[0] = \vec{x}_0 \tag{2}$$

or a differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \tag{3}$$
$$\vec{x}(0) = \vec{x}_0. \tag{4}$$

However, many systems in the real world can only be faithfully represented by *nonlinear* models. A very nonexhaustive list of such models follows.

1. Up until now, we have considered transistors to be binary – turning off or on at some voltage differential. But in reality, transistors work continuously, and the relevant governing equations are highly nonlinear. For example, the Wikipedia article on BJTs has some of them.
2. Robotics and control in general has highly nonlinear dynamics. The reason for this is because the system has a variety of geometric relationships that must hold, and so there are many trigonometric functions involved, which are all non-linear but have good linear approximations. An example of this analysis may be found when applying control to stabilize an inverted pendulum on a cart.
3. Machine learning and optimization also have highly nonlinear systems. In particular, the continuous-time concept of *gradient flow* and the discrete-time concept of *gradient descent* result in highly nonlinear state trajectories.

More generally, a nonlinear model takes the generic form of a difference equation or a differential equation, which we formally define here for reference later.

Model 1 (Discrete-Time Time-Invariant Difference Equation Model)
The model is of the form

$$\vec{x}[i+1] = \vec{f}(\vec{x}[i], \vec{u}[i]) \tag{5}$$
$$\vec{x}[0] = \vec{x}_0 \tag{6}$$

for $\vec{x}\colon \mathbb{N} \to \mathbb{R}^n$ the state vector as a function of timestep, $\vec{u}\colon \mathbb{N} \to \mathbb{R}^m$ the control inputs as a function of timestep, and $\vec{f}\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ a function.

Model 2 (Continuous-Time Time-Invariant Differential Equation Model)
The model is of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \vec{f}(\vec{x}(t), \vec{u}(t)) \tag{7}$$

$$\vec{x}(0) = \vec{x}_0 \tag{8}$$

for $\vec{x}\colon \mathbb{R}_+ \to \mathbb{R}^n$ the state vector as a function of timestep, $\vec{u}\colon \mathbb{R}_+ \to \mathbb{R}^m$ the control inputs as a function of timestep, and $\vec{f}\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ a function.

We will learn how to approximate these models locally by linear models, in a process called *linearization* of the function $\vec{f}$.

Key Idea 3 (Local Linearization)
Linearization of a function is the technique of approximating it by a linear function "locally" (i.e., in a small region around some point).

In Section 2, we will learn what a local linearization of a function is, and how to compute it. In Section 3 we will show how to linearize models and use linearized models for control. We conclude with some real-world models that we linearize in Section 4.

2 Linearization

NOTE: This section will have some definitions (for derivatives) which look abstract initially, but have relatively simple formulas. This is done because the abstract definition is used to give a geometric viewpoint, which makes the process of finding the "correct" approximation relatively simple. Then, to actually compute the correct approximation, we look at its vector/matrix representation in the standard basis, giving us a concrete formula that is the one we are going to use 99% of the time.

So, don't freak out if the details look technical. Just try to understand the overall picture, and then you can read the formulas, which are most of what you will need.

2.1 Linear Approximations

The key to linearization is the *first derivative* concept. Recall the familiar limit definition of the derivative for a function $f\colon \mathbb{R} \to \mathbb{R}$, i.e., f is differentiable at $x^\star$ with derivative $f'(x^\star)$ if and only if the following limit exists and equality holds:

$$f'(x^\star) = \lim_{x \to x^\star} \frac{f(x) - f(x^\star)}{x - x^\star}. \tag{9}$$

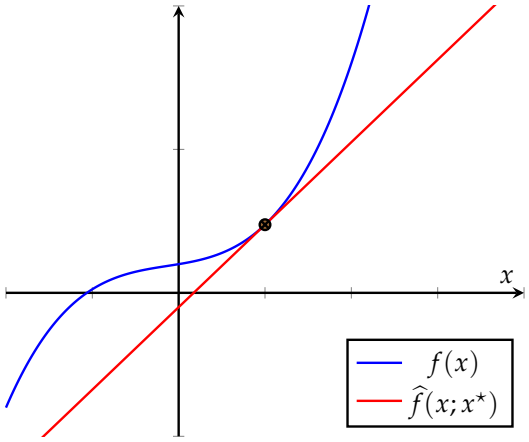
Rearranging, we say that f is differentiable at $x^\star$ with derivative $f'(x^\star)$, if and only if the following limit exists and equality holds:

$$0 = \lim_{x \to x^\star} \frac{f(x) - [f(x^\star) + f'(x^\star) \cdot (x - x^\star)]}{x - x^\star} \tag{10}$$

The bracketed quantity

$$\widehat{f}(x; x^\star) := f(x^\star) + f'(x^\star) \cdot (x - x^\star) \tag{11}$$

is exactly the linearization, i.e., linear approximation of f around $x^\star$. The derivative eq. (10) says that for x very close to $x^\star$, the linear approximation $\widehat{f}(x;x^\star)$ is almost exactly $f(x)$.



This definition of the first derivative – i.e., an intercept $f(x^\star)$ plus a linear function of $x - x^\star$ representing the tangent plane – is actually the correct one for generalization to vector functions. In fact, we give the definition in full generality now, and it will not look much different.

Definition 4 (First Derivative, Jacobian)
Let $\vec{f}\colon \mathbb{R}^p \to \mathbb{R}^q$ be a function. We say that $\vec{f}$ is *differentiable* at $\vec{x}^\star \in \mathbb{R}^p$ with derivative $J\vec{f}(\vec{x}^\star) \in \mathbb{R}^{q\times p}$ if and only if the following limit exists and equality holds:

$$0 = \lim_{\vec{x}\to\vec{x}^\star} \frac{\left\|\vec{f}(\vec{x}) - \left[\vec{f}(\vec{x}^\star) + J\vec{f}(\vec{x}^\star)\cdot(\vec{x}-\vec{x}^\star)\right]\right\|}{\|\vec{x}-\vec{x}^\star\|} \tag{12}$$

We call $J\vec{f}(\vec{x}^\star)$ the *Jacobian* or *derivative* of $\vec{f}$ at $\vec{x}^\star$.
More formally, we define the Jacobian $J\vec{f}\colon \mathbb{R}^p \to \mathbb{R}^{q\times p}$ as the matrix-valued function which takes in points $\vec{x}^\star \in \mathbb{R}^p$ and outputs derivative matrices $J\vec{f}(\vec{x}^\star) \in \mathbb{R}^{q\times p}$.
We say that $\vec{f}$ is *differentiable* if it is differentiable at every point $\vec{x}^\star \in \mathbb{R}^p$.

The bracketed quantity

$$\vec{\widehat{f}}(\vec{x};\vec{x}^\star) := \vec{f}(\vec{x}^\star) + J\vec{f}(\vec{x}^\star)\cdot(\vec{x}-\vec{x}^\star) \tag{13}$$

is exactly the best linear approximation of $\vec{f}$ around $\vec{x}^\star$. The derivative eq. (12) says that for $\vec{x}$ very close to $\vec{x}^\star$, the linear approximation $\vec{\widehat{f}}(\vec{x};\vec{x}^\star)$ is almost exactly $\vec{f}(\vec{x})$.

This motivates the following definition of the best linear approximation, i.e., the linearization.

Definition 5 (Linearization)
Suppose $\vec{f}\colon \mathbb{R}^p \to \mathbb{R}^q$ is differentiable. Then the *linearization* of $\vec{f}$ around $\vec{x}^\star \in \mathbb{R}^p$ is the function $\vec{\widehat{f}}(\cdot;\vec{x}^\star)\colon \mathbb{R}^p \to \mathbb{R}^q$ given by

$$\vec{\widehat{f}}(\vec{x};\vec{x}^\star) := \vec{f}(\vec{x}^\star) + J\vec{f}(\vec{x}^\star)\cdot(\vec{x}-\vec{x}^\star). \tag{14}$$

2.2 Computing the Derivative

In Definition 4, we defined our first derivative for multivariable functions, analogously to the single-variable derivative definition in Equation (10). But, just like in the single-variable case, if all we could use to compute the derivative is that specific definition, we would get stuck. Now, we introduce a much more mechanical and easier way to compute the derivative.

The main building block is the *partial derivative*, which is defined almost analogously to a single-variable derivative.

Definition 6 (Partial Derivative)
Let $f: \mathbb{R}^p \to \mathbb{R}$ be differentiable. The *partial derivative of f with respect to x_i* is the function $\frac{\partial f}{\partial x_i}: \mathbb{R}^p \to \mathbb{R}$ defined by

$$\frac{\partial f}{\partial x_i}(\vec{x}) := \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_p) - f(x_1, \ldots, x_p)}{h}. \tag{15}$$

This definition provides a way to compute the partial derivative.

Key Idea 7 (Computing Partial Derivatives)
To compute a partial derivative $\frac{\partial f}{\partial x_i}$:

- Write out f explicitly in terms of $x_1, \ldots, x_p$.
- Pretend all variables x_j are actually constants, *except* the variable x_i.
- Take the single-variable derivative of f in x_i.

The result will be the function $\frac{\partial f}{\partial x_i}$.

We can now give an explicit formula for the Jacobian.

Theorem 8 (Jacobian in terms of Partial Derivatives)
Let $\vec{f}: \mathbb{R}^p \to \mathbb{R}^q$ be differentiable at $\vec{x}$. Then the Jacobian at $\vec{x}$, $J\vec{f}(\vec{x})$, is uniquely given by

$$J\vec{f}(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}) & \cdots & \frac{\partial f_1}{\partial x_p}(\vec{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_q}{\partial x_1}(\vec{x}) & \cdots & \frac{\partial f_q}{\partial x_p}(\vec{x}) \end{bmatrix} \tag{16}$$

Here "uniquely" means that there is no other matrix, written in the standard basis, which represents the Jacobian. Just like a differentiable function $f: \mathbb{R} \to \mathbb{R}$ has exactly one derivative at a given point, a differentiable function $\vec{f}: \mathbb{R}^p \to \mathbb{R}^q$ has exactly one derivative at a given point. And so we can get an *unambiguously best linearization* using the Jacobian.

A formal proof of Theorem 8 is out of scope of the course. However, the main idea can be obtained by only considering polynomial functions $f: \mathbb{R}^p \to \mathbb{R}$. For instance, consider the polynomial $f(x_1, x_2) := (ax_1 + bx_2)^2$. By the chain rule (which follows from the chain rule for single-variable derivatives), we have

$$\frac{\partial f}{\partial x_1}(x_1, x_2) = 2a(ax_1 + bx_2) \qquad \frac{\partial f}{\partial x_2}(x_1, x_2) = 2b(ax_1 + bx_2). \tag{17}$$

Fix a point $\vec{x}^\star \in \mathbb{R}^2$ and a "perturbed" vector $\vec{x}$ which is close to $\vec{x}^\star$. Then

$$\begin{aligned}
f(\vec{x}) &= (ax_1 + ax_2)^2 && (18)\\
&= (ax_1^\star + a(x_1 - x_1^\star) + bx_2^\star + b(x_2 - x_2^\star))^2 && (19)\\
&= a^2x_1^{\star 2} + 2abx_1^\star x_2^\star + b^2x_2^{\star 2} && (20)\\
&\quad + 2a(ax_1^\star + bx_2^\star)(x_1 - x_1^\star) + 2b(ax_1^\star + bx_2^\star)(x_2 - x_2^\star) && (21)\\
&\quad + a^2(x_1 - x_1^\star)^2 + 2ab(x_1 - x_1^\star)(x_2 - x_2^\star) + b^2(x_2 - x_2^\star)^2 && (22)\\
&= f(x_1^\star, x_2^\star) + \frac{\partial f}{\partial x_1}(x_1^\star, x_2^\star) \cdot (x_1 - x_1^\star) + \frac{\partial f}{\partial x_2}(x_1^\star, x_2^\star) \cdot (x_2 - x_2^\star) && (23)\\
&\quad + a^2(x_1 - x_1^\star)^2 + 2ab(x_1 - x_1^\star)(x_2 - x_2^\star) + b^2(x_2 - x_2^\star)^2. && (24)
\end{aligned}$$

Now since $\vec{x} - \vec{x}^\star$ is small, the quantities $x_i - x_i^\star$ are small. Thus the quantities $(x_i - x_i^\star)^2$ and $(x_i - x_i^\star)(x_j - x_j^\star)$ are even smaller, so the whole last line is negligible and it is reasonable to consider the linear approximation

$$f(\vec{x}) \approx \widehat{f}(\vec{x}; \vec{x}^\star) := f(x_1^\star, x_2^\star) + \frac{\partial f}{\partial x_1}(x_1^\star, x_2^\star) \cdot (x_1 - x_1^\star) + \frac{\partial f}{\partial x_2}(x_1^\star, x_2^\star) \cdot (x_2 - x_2^\star). \tag{25}$$

Now this quantity can be expressed using the Jacobian, whose formula was explicitly given in Theorem 8.

$$\widehat{f}(\vec{x}; \vec{x}^\star) = f(x_1^\star, x_2^\star) + \begin{bmatrix} \frac{\partial f}{\partial x_1}(x_1^\star, x_2^\star) & \frac{\partial f}{\partial x_2}(x_1^\star, x_2^\star) \end{bmatrix} \begin{bmatrix} x_1 - x_1^\star \\ x_2 - x_2^\star \end{bmatrix} = f(\vec{x}^\star) + Jf(x_1^\star, x_2^\star) \cdot (\vec{x} - \vec{x}^\star) \tag{26}$$

which is exactly the linearization formula given in Definition 5.

2.3 (OPTIONAL) Higher-Order Partial Derivatives and Quadratic Approximation

So far we have learned how to linearize functions around points. This is good, and useful for many cases. However, sometimes there is some kind of "essential nonlinearity" of our function that we would like to preserve in our approximation. This scenario is most common[1] with scalar-valued functions , so we consider only functions $f : \mathbb{R}^p \to \mathbb{R}$. In this cases, we can turn to *quadratic approximation*.[2]

Just as first derivatives were the key tool for linearization, second derivatives will be the key tool for quadratic approximation. In fact, we can get the best quadratic approximation straight from the definition of the second derivative.

Definition 9 (Second Derivative, Hessian)
Let $f : \mathbb{R}^p \to \mathbb{R}$ be differentiable at $\vec{x}^\star \in \mathbb{R}^p$ with derivative Jf. We say that f is *twice differentiable* at $\vec{x}^\star \in \mathbb{R}^p$ with second derivative $Hf(\vec{x}^\star) \in \mathbb{R}^{p \times p}$ if and only if the following limit exists and equality holds:

$$0 = \lim_{\vec{x} \to \vec{x}^\star} \frac{\left\| \vec{f}(\vec{x}) - \left[\vec{f}(\vec{x}^\star) + Jf(\vec{x}^\star) \cdot (\vec{x} - \vec{x}^\star) + \frac{1}{2} \langle Hf(\vec{x}^\star) \cdot (\vec{x} - \vec{x}^\star),\ \vec{x} - \vec{x}^\star \rangle \right] \right\|}{\|\vec{x} - \vec{x}^\star\|^2} \tag{27}$$

We call $Hf(\vec{x}^\star)$ the *Hessian* or *second derivative* of f at $\vec{x}^\star$.
More formally, we define the Hessian $Hf : \mathbb{R}^p \to \mathbb{R}^{p \times p}$ as the matrix-valued function which takes in points $\vec{x}^\star \in \mathbb{R}^p$ and outputs second derivative matrices $Hf(\vec{x}^\star) \in \mathbb{R}^{p \times p}$.

[1] In optimization, signal processing, and machine learning, to name a few.
[2] Quadratic approximation is also possible for functions $\vec{f} : \mathbb{R}^p \to \mathbb{R}^q$, but this requires the introduction of *tensors* as a concept, which is too far out of scope of this class.

We say that f is *twice differentiable* if it is twice differentiable at every point $\vec{x}^\star \in \mathbb{R}^p$.

This definition is slightly atypical and perhaps not what we expected. But it all clears up when we note that functions of the type $a + \left\langle \vec{b},\ \vec{x} - \vec{x}^\star \right\rangle + \langle C(\vec{x} - \vec{x}^\star),\ \vec{x} - \vec{x}^\star \rangle$ are exactly the quadratic functions $\mathbb{R}^p \to \mathbb{R}$. And so the definition of the second derivative just fits the best quadratic approximation to f at $\vec{x}^\star$.

There are a couple of quirks here. First, it is probably not clear that the third term inner product, i.e., the term $\langle Hf(\vec{x}^\star) \cdot (\vec{x} - \vec{x}^\star),\ \vec{x} - \vec{x}^\star \rangle$ is actually a quadratic function. But if we do the matrix-vector multiplication and take the inner product, we have

$$\langle Hf(\vec{x}^\star) \cdot (\vec{x} - \vec{x}^\star),\ \vec{x} - \vec{x}^\star \rangle = \sum_{i=1}^{p} \sum_{j=1}^{p} (Hf(\vec{x}^\star))_{ij} (\vec{x} - \vec{x}^\star)_i (\vec{x} - \vec{x}^\star)_j \tag{28}$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{p} (Hf(\vec{x}^\star))_{ij} (x_i - x_i^\star)(x_j - x_j^\star) \tag{29}$$

which is a quadratic polynomial in each variable $x_1, \ldots, x_p$. Also, the term $\frac{1}{2}$ in front of this inner product in Equation (27) doesn't need to be there (we could just scale the derivative by 2). It is there because it means that the Hessian has a nicer representation in terms of the second-order partial derivatives. Third, we divide by $\|\vec{x} - \vec{x}^\star\|^2$ because if we only divided by $\|\vec{x} - \vec{x}^\star\|$ (as in the case for the first derivative), then any matrix at all would be a suitable second derivative at any point. More precisely, let $M \in \mathbb{R}^{p \times p}$ be any matrix; then we have

$$\frac{\langle M(\vec{x} - \vec{x}^\star),\ \vec{x} - \vec{x}^\star \rangle}{\|\vec{x} - \vec{x}^\star\|} = \left\langle M \underbrace{\frac{\vec{x} - \vec{x}^\star}{\|\vec{x} - \vec{x}^\star\|}}_{\text{unit norm}},\ \underbrace{\vec{x} - \vec{x}^\star}_{\text{small}, \to \vec{0}_p} \right\rangle \to 0 \tag{30}$$

as $\vec{x} \to \vec{x}^\star$, and so the quadratic function itself would go to 0 and be irrelevant. But

$$\frac{\langle M(\vec{x} - \vec{x}^\star),\ \vec{x} - \vec{x}^\star \rangle}{\|\vec{x} - \vec{x}^\star\|^2} = \left\langle M \underbrace{\frac{\vec{x} - \vec{x}^\star}{\|\vec{x} - \vec{x}^\star\|}}_{\text{unit norm}},\ \underbrace{\frac{\vec{x} - \vec{x}^\star}{\|\vec{x} - \vec{x}^\star\|}}_{\text{unit norm}} \right\rangle \tag{31}$$

which doesn't necessarily go to 0 as $\vec{x} \to \vec{x}^\star$. So the quadratic function is relevant and defines a derivative.

This gives us the best quadratic approximation.

Definition 10 (Best Quadratic Approximation)
Suppose $f\colon \mathbb{R}^p \to \mathbb{R}$ is twice differentiable. Then the *best quadratic approximation* of f around $\vec{x}^\star \in \mathbb{R}^p$ is the function $\widehat{f}_{\text{quad}}(\cdot; \vec{x}^\star)\colon \mathbb{R}^p \to \mathbb{R}$ given by

$$\widehat{f}_{\text{quad}}(\vec{x}; \vec{x}^\star) := f(\vec{x}^\star) + Jf(\vec{x}^\star) \cdot (\vec{x} - \vec{x}^\star) + \frac{1}{2} \langle Hf(\vec{x}^\star) \cdot (\vec{x} - \vec{x}^\star),\ \vec{x} - \vec{x}^\star \rangle . \tag{32}$$

Again, we construct an explicit formula for the Hessian out of partial derivatives. For that, we require a notion of second partial derivatives.

Definition 11 (Second Partial Derivatives)
Let $f: \mathbb{R}^p \to \mathbb{R}$ be twice differentiable. The *second partial derivative of f with respect to x_i and x_j* is the function $\frac{\partial^2 f}{\partial x_i \partial x_j}: \mathbb{R}^p \to \mathbb{R}$ defined by

$$\frac{\partial^2 f}{\partial x_i \partial x_j} := \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right). \tag{33}$$

Note that we can take the second partial derivative with respect to x_i twice. The notation for this is $\frac{\partial^2 f}{\partial x_i^2}$.

Fortunately, most of the time the order of partial derivatives does not matter, thanks to the following result.

Theorem 12 (Clairaut's Theorem)
Let $f: \mathbb{R}^p \to \mathbb{R}$ be twice differentiable, and all second partial derivatives of f be continuous. Then for any i and j,

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}, \tag{34}$$

i.e., the order of taking partial derivatives is irrelevant.

Now we can give a formula for the Hessian.

Theorem 13 (Hessian in terms of Partial Derivatives)
Let $f: \mathbb{R}^p \to \mathbb{R}$ be twice differentiable at $\vec{x}$. Then the Hessian at $\vec{x}$, $Hf(\vec{x})$, is uniquely given by

$$Hf(\vec{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\vec{x}) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\vec{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\vec{x}) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\vec{x}) \end{bmatrix}. \tag{35}$$

Again, we cannot prove this theorem. However, we may look at it through the example earlier, i.e., $f(x_1, x_2) = (ax_1 + bx_2)^2$. In this example, we have the second partial derivatives

$$\frac{\partial^2 f}{\partial x_1^2}(x_1, x_2) = 2a^2 \qquad \frac{\partial^2 f}{\partial x_2^2}(x_1, x_2) = 2b^2 \qquad \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1, x_2) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_1, x_2) = 2ab. \tag{36}$$

Thus the Hessian is given by

$$Hf(x_1, x_2) = \begin{bmatrix} 2a^2 & 2ab \\ 2ab & 2b^2 \end{bmatrix}. \tag{37}$$

Recall that for $\vec{x}, \vec{x}^\star \in \mathbb{R}^2$, we computed

$$f(\vec{x}) = f(\vec{x}^\star) + Jf(\vec{x}^\star) \cdot (\vec{x} - \vec{x}^\star) + a^2(x_1 - x_1^\star)^2 + 2ab(x_1 - x_1^\star)(x_2 - x_2^\star) + b^2(x_2 - x_2^\star)^2. \tag{38}$$

We may relate this to the Hessian; we have

$$f(\vec{x}) = f(\vec{x}^\star) + Jf(\vec{x}^\star) \cdot (\vec{x} - \vec{x}^\star) + a^2(x_1 - x_1^\star)^2 + 2ab(x_1 - x_1^\star)(x_2 - x_2^\star) + b^2(x_2 - x_2^\star)^2 \tag{39}$$

$$= f(\vec{x}^\star) + Jf(\vec{x}^\star) \cdot (\vec{x} - \vec{x}^\star) + \frac{1}{2}\begin{bmatrix} x_1 - x_1^\star & x_2 - x_2^\star \end{bmatrix} \begin{bmatrix} 2a^2 & 2ab \\ 2ab & 2b^2 \end{bmatrix} \begin{bmatrix} x_1 - x_1^\star \\ x_2 - x_2^\star \end{bmatrix} \tag{40}$$

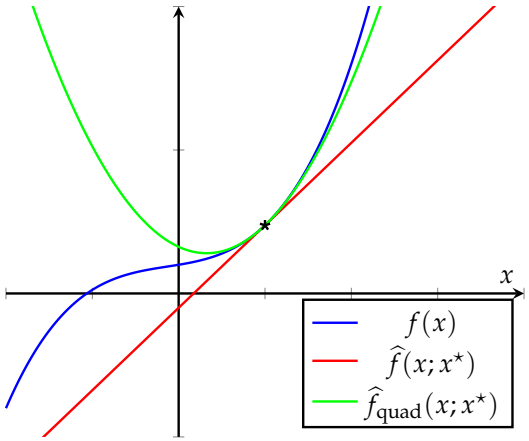
$$= f(\vec{x}^\star) + Jf(\vec{x}^\star) \cdot (\vec{x} - \vec{x}^\star) + \frac{1}{2}(\vec{x} - \vec{x}^\star)^\top (Hf(\vec{x}^\star))(\vec{x} - \vec{x}^\star) \tag{41}$$

$$= f(\vec{x}^\star) + Jf(\vec{x}^\star) \cdot (\vec{x} - \vec{x}^\star) + \frac{1}{2}\langle Hf(\vec{x}^\star) \cdot (\vec{x} - \vec{x}^\star),\ \vec{x} - \vec{x}^\star \rangle \tag{42}$$

$$= \widehat{f}_{\text{quad}}(\vec{x}; \vec{x}^\star) \tag{43}$$

which is the same formula as Definition 10. As another verification, f is itself a quadratic function, so the best quadratic approximation to f at $\vec{x}^\star$ should be $\widehat{f}_{\text{quad}}(\cdot; \vec{x}^\star) = f$ itself. And indeed it is.

Finally, here is a graphical comparison of the best quadratic approximation with the linear approximation from earlier.



3 Linearizing Control Models

Recall that we originally wanted to linearize state update functions $\vec{f}\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, that either took in a state and input pair $(\vec{x}, \vec{u})$, and output either a derivative $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t)$ or a new state $\vec{x}[i+1]$. To do this linearization, we introduce some new notation for the Jacobian.

Notation 14

Let $\vec{f}\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be differentiable. We define the Jacobian with respect to $\vec{x}$, i.e., $J_{\vec{x}}\vec{f}\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times n}$, and with respect to $\vec{u}$, i.e., $J_{\vec{u}}\vec{f}\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times m}$, as being given by

$$J_{\vec{x}}\vec{f}(\vec{x}, \vec{u}) := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}, \vec{u}) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{x}, \vec{u}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\vec{x}, \vec{u}) & \cdots & \frac{\partial f_n}{\partial x_n}(\vec{x}, \vec{u}) \end{bmatrix} \tag{44}$$

$$J_{\vec{u}}\vec{f}(\vec{x}, \vec{u}) := \begin{bmatrix} \frac{\partial f_1}{\partial u_1}(\vec{x}, \vec{u}) & \cdots & \frac{\partial f_1}{\partial u_m}(\vec{x}, \vec{u}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1}(\vec{x}, \vec{u}) & \cdots & \frac{\partial f_n}{\partial u_m}(\vec{x}, \vec{u}) \end{bmatrix}. \tag{45}$$

Under this notation, we have the following linearization of the state update function $\vec{f}$.

Key Idea 15 (Linearization of State Update Function)
Let $\vec{f}\colon \mathbb{R}^n \times \mathbb{R}^m$ be differentiable. The linearization of $\vec{f}$ around $(\vec{x}^\star, \vec{u}^\star)$ is given by

$$\hat{\vec{f}}(\vec{x}, \vec{u}; \vec{x}^\star, \vec{u}^\star) := \vec{f}(\vec{x}^\star, \vec{u}^\star) + J_{\vec{x}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot (\vec{x} - \vec{x}^\star) + J_{\vec{u}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot (\vec{u} - \vec{u}^\star). \tag{46}$$

3.1 Linearization of Discrete-Time Time-Invariant Difference Equation Model

Recall that in the Discrete-Time Time-Invariant Difference Equation Model, the update rule is

$$\vec{x}[i+1] = \vec{f}(\vec{x}[i], \vec{u}[i]). \tag{47}$$

One may linearize $\vec{f}$, but in order to do this, we need to pick a point $(\vec{x}^\star, \vec{u}^\star)$ to linearize around. We would like the system dynamics to be well-behaved around $(\vec{x}^\star, \vec{u}^\star)$. This leads to the concept of *equilibrium point*.

Definition 16 (Equilibrium Point in Discrete-Time Time-Invariant Difference Equation Model)
In the Discrete-Time Time-Invariant Difference Equation Model, the point $(\vec{x}^\star, \vec{u}^\star)$ is an *equilibrium point* if and only if

$$\vec{f}(\vec{x}^\star, \vec{u}^\star) = \vec{x}^\star. \tag{48}$$

In certain contexts this is also called an *operating point*.

Intuitively, $(\vec{x}^\star, \vec{u}^\star)$ can be interpreted as a stationary point of the system dynamics, in the following sense: *if the state starts at $\vec{x}^\star$, and input $\vec{u}^\star$ is applied at every timestep, the state never leaves $\vec{x}^\star$.*

Linearizing the right-hand side around some equilibrium point $(\vec{x}^\star, \vec{u}^\star)$ which is close to $(\vec{x}[i], \vec{u}[i])$, we have

$$\begin{aligned}
\vec{x}[i+1] &= \vec{f}(\vec{x}[i], \vec{u}[i]) && (49)\\
&\approx \hat{\vec{f}}(\vec{x}[i], \vec{u}[i]; \vec{x}^\star, \vec{u}^\star) && (50)\\
&= \vec{f}(\vec{x}^\star, \vec{u}^\star) + J_{\vec{x}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot (\vec{x}[i] - \vec{x}^\star) + J_{\vec{u}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot (\vec{u}[i] - \vec{u}^\star) && (51)\\
&= \vec{x}^\star + J_{\vec{x}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot (\vec{x}[i] - \vec{x}^\star) + J_{\vec{u}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot (\vec{u}[i] - \vec{u}^\star). && (52)
\end{aligned}$$

If we define the *deviation from the equilibrium point*

$$\delta\vec{x}[i] := \vec{x}[i] - \vec{x}^\star \qquad \delta\vec{u}[i] := \vec{u}[i] - \vec{u}^\star, \tag{53}$$

then the linearized equation becomes

$$\delta\vec{x}[i+1] = J_{\vec{x}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot \delta\vec{x}[i] + J_{\vec{u}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot \delta\vec{u}[i] \tag{54}$$

is a linear system in $\delta\vec{x}[i]$. We can thus analyze its stability, controllability, and so on, using the tools we already developed. We can even do feedback control! Keep in mind that in this system, driving $\delta\vec{x}[i]$ to $\vec{0}_n$ (for instance via feedback stabilization) is equivalent to driving $\vec{x}[i]$ to $\vec{x}^\star$. So, one thing we can do is to have $\vec{x}^\star$ be the state we want to drive the model to, and then use feedback control to get there.

When we're done, we can go back to the "real" state by setting $\vec{x}[i] = \delta\vec{x}[i] + \vec{x}^\star$.

We summarize our findings below.

Proposition 17 (Linearizing the Discrete-Time Time-Invariant Difference Equation Model)
Suppose $(\vec{x}^\star, \vec{u}^\star)$ is an equilibrium point of the Discrete-Time Time-Invariant Difference Equation Model. Define

$$\delta\vec{x}[i] := \vec{x}[i] - \vec{x}^\star \qquad \delta\vec{u}[i] := \vec{u}[i] - \vec{u}^\star. \tag{55}$$

Then a linearization of Discrete-Time Time-Invariant Difference Equation Model results in the linear model

$$\delta\vec{x}[i+1] = J_{\vec{x}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot \delta\vec{x}[i] + J_{\vec{u}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot \delta\vec{u}[i] \tag{56}$$

for $\delta\vec{x}[i]$ and $\delta\vec{u}[i]$ very small (i.e., $\vec{x}[i] \approx \vec{x}^\star$ and $\vec{u}[i] \approx \vec{u}^\star$).

Warning 18
The linearization is *only* valid when the state-input pair $(\vec{x}[i], \vec{u}[i])$ is contained in a small neighborhood of the equilibrium point $(\vec{x}^\star, \vec{u}^\star)$.

3.2 Linearization of Continuous-Time Time-Invariant Differential Equation Model

The analysis of the continuous-time goes similarly to the discrete time, with a couple of crucial differences.

Recall that in the Continuous-Time Time-Invariant Differential Equation Model, the update rule is

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \vec{f}(\vec{x}(t), \vec{u}(t)). \tag{57}$$

Again, we need to define *equilibrium point*.

Definition 19 (Equilibrium Point in Continuous-Time Time-Invariant Differential Equation Model)
In the Continuous-Time Time-Invariant Differential Equation Model, the point $(\vec{x}^\star, \vec{u}^\star)$ is an *equilibrium point* if and only if

$$\vec{f}(\vec{x}^\star, \vec{u}^\star) = \vec{0}_n. \tag{58}$$

In certain contexts this is also called an *operating point*.

Intuitively, $(\vec{x}^\star, \vec{u}^\star)$ can be interpreted as a stationary point of the system dynamics, in the following sense: *if the state starts at $\vec{x}^\star$, and input $\vec{u}^\star$ is applied at every time, the state never leaves $\vec{x}^\star$.*

Linearizing the right-hand side around some equilibrium point $(\vec{x}^\star, \vec{u}^\star)$ which is close to $(\vec{x}(t), \vec{u}(t))$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) &= \vec{f}(\vec{x}(t), \vec{u}(t)) && (59)\\
&\approx \vec{\hat{f}}(\vec{x}(t), \vec{u}(t); \vec{x}^\star, \vec{u}^\star) && (60)\\
&= \vec{f}(\vec{x}^\star, \vec{u}^\star) + J_{\vec{x}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot (\vec{x}(t) - \vec{x}^\star) + J_{\vec{u}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot (\vec{u}(t) - \vec{u}^\star) && (61)\\
&= J_{\vec{x}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot (\vec{x}(t) - \vec{x}^\star) + J_{\vec{u}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot (\vec{u}(t) - \vec{u}^\star). && (62)
\end{aligned}$$

If we define the *deviation from the equilibrium point*

$$\delta\vec{x}(t) := \vec{x}(t) - \vec{x}^\star \qquad \delta\vec{u}(t) := \vec{u}(t) - \vec{u}^\star, \tag{63}$$

then the linearized equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{x}(t) \approx J_{\vec{x}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot \delta\vec{x}(t) + J_{\vec{u}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot \delta\vec{u}(t) \tag{64}$$

is a linear system in $\delta\vec{x}(t)$. We can thus analyze its stability, controllability, and so on, using the tools we already developed. We can even do feedback control! Keep in mind that in this system, driving $\delta\vec{x}(t)$ to $\vec{0}_n$ (for instance via feedback stabilization) is equivalent to driving $\vec{x}(t)$ to $\vec{x}^\star$. So, one thing we can do is to have $\vec{x}^\star$ be the state we want to drive the model to, and then use feedback control to get there.

When we're done, we can go back to the "real" state by setting $\vec{x}(t) = \delta\vec{x}(t) + \vec{x}^\star$.

We summarize our findings below.

Proposition 20 (Linearizing the Continuous-Time Time-Invariant Differential Equation Model)
Suppose $(\vec{x}^\star, \vec{u}^\star)$ is an equilibrium point of the Continuous-Time Time-Invariant Differential Equation Model. Define

$$\delta\vec{x}(t) := \vec{x}(t) - \vec{x}^\star \qquad \delta\vec{u}(t) := \vec{u}(t) - \vec{u}^\star. \tag{65}$$

Then a linearization of Continuous-Time Time-Invariant Differential Equation Model results in the linear model

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{x}(t) = J_{\vec{x}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot \delta\vec{x}(t) + J_{\vec{u}}\vec{f}(\vec{x}^\star, \vec{u}^\star) \cdot \delta\vec{u}(t) \tag{66}$$

for $\delta\vec{x}(t)$ and $\delta\vec{u}(t)$ very small (i.e., $\vec{x}(t) \approx \vec{x}^\star$ and $\vec{u}(t) \approx \vec{u}^\star$).

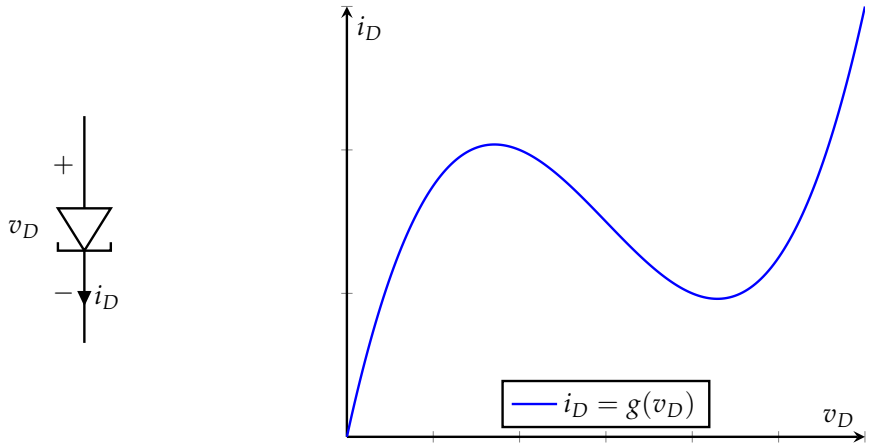
Warning 21
The linearization is *only* valid when the state-input pair $(\vec{x}(t), \vec{u}(t))$ is contained in a small neighborhood of the equilibrium point $(\vec{x}^\star, \vec{u}^\star)$.

4 Examples

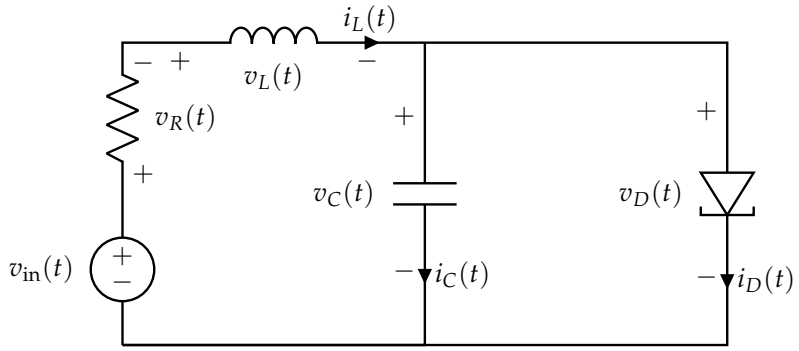
Most examples "in the wild" are in continuous-time, since they are motivated by physics. We could discretize our continuous-time nonlinear models and get discrete-time nonlinear models, however.

4.1 Circuits Example

While we could discuss the nonlinear BJT model we alluded to, that model is quite complicated and thus not suitable for an example. Instead we will discuss the case of a *tunnel diode* model. A tunnel diode is characterized by an I-V relationship where, for a certain voltage range, the current decreases with increasing voltage. (This is due to a quantum mechanical effect called *tunneling*).



Now consider the circuit below:



Using KVL and KCL, and the fact that $i_D(t) = g(v_D(t))$, we get the state model

$$\frac{\mathrm{d}}{\mathrm{d}t}v_C(t) = -\frac{1}{C}g(v_C(t)) + \frac{1}{C}i_L(t) \tag{67}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}i_L(t) = -\frac{1}{L}v_C(t) + \frac{R}{L}i_L(t) - \frac{1}{L}v_{\text{in}}(t). \tag{68}$$

Thus $\vec{f}$ is given by

$$\vec{f}(\underbrace{v_C, i_L}_{=\vec{x}}, \underbrace{v_{\text{in}}}_{=u}) = \begin{bmatrix} -\frac{1}{C}g(v_C) + \frac{1}{C}i_L \\ -\frac{1}{L}v_C + \frac{R}{L}i_L - \frac{1}{L}v_{\text{in}} \end{bmatrix}. \tag{69}$$

To find an equilibrium point, we set $\vec{f}(v_C^\star, i_L^\star, v_{\text{in}}^\star)$ to 0 (since this is an instance of Continuous-Time Time-Invariant Differential Equation Model) and solve for $v_C^\star$, $i_L^\star$, and $v_{\text{in}}^\star$. Indeed, we have the system of equations

$$0 = -\frac{1}{C}g(v_C(t)) + \frac{1}{C}i_L(t) \tag{70}$$

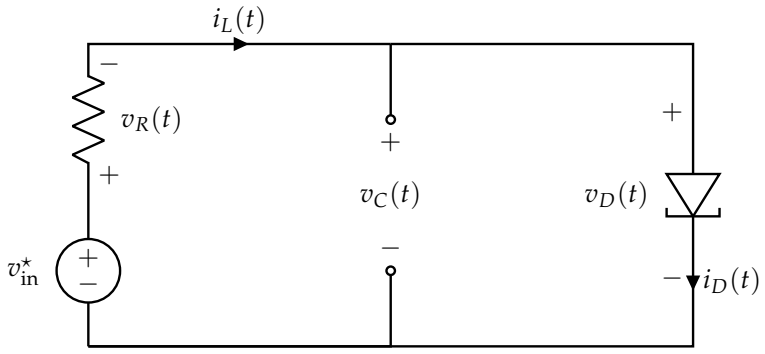
$$0 = -\frac{1}{L}v_C(t) + \frac{R}{L}i_L(t) - \frac{1}{L}v_{\text{in}}(t). \tag{71}$$

Solving, we get that the equilibrium point $(v_C^\star, i_L^\star, v_{\text{in}}^\star)$ is any triple which satisfies the equations

$$i_L^\star = g(v_C^\star) \tag{72}$$

$$i_L^\star = \frac{v_C^\star + v_{\text{in}}^\star}{R}. \tag{73}$$

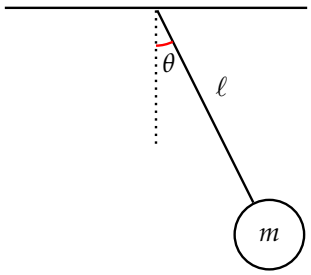
To get further insight into equilibrium states of circuits, note that to solve for the equilibrium we set $\frac{\mathrm{d}}{\mathrm{d}t}v_C(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}i_L(t)$ to 0. Since $i_C(t) = C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t}$ and $v_L(t) = L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t}$, we thus have that at equilibrium, $i_C(t) = 0$ and $v_L(t) = 0$. Thus at equilibrium, the capacitor acts like an open circuit and the inductor like a short circuit. Redrawing the circuit, this is the picture at equilibrium:



We can linearize this new, simplified system to analyze small perturbations to v_C, i_L, v_{in} from the equilibrium states $v_C^\star, i_L^\star, v_{\text{in}}^\star$.

4.2 Mechanics Example

Consider the following pendulum with mass m:



From physics we know that the equation of motion of this pendulum is governed by the differential equation

$$m\ell\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} = -k\ell\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} - mg\sin(\theta(t)) \tag{74}$$

where k is some air resistance coefficient. We define the state space variables

$$x_1(t) := \theta(t) \qquad x_2(t) := \frac{\mathrm{d}}{\mathrm{d}t}\theta(t). \tag{75}$$

This gives the nonlninear system

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = x_2(t) \tag{76}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = -\frac{\mathrm{g}}{\ell}\sin(x_1(t)) - \frac{k}{m}x_2(t) \tag{77}$$

where g is the gravitational acceleration. Thus $\vec{f}$ is given by

$$\vec{f}(x_1, x_2) = \begin{bmatrix} x_2 \\ -\frac{\mathrm{g}}{\ell}\sin(x_1) - \frac{k}{m}x_2 \end{bmatrix}. \tag{78}$$

There are two distinct equilibrium points, which we get from setting $\vec{f}(x_1^\star, x_2^\star)$ to 0 and solving:

$$(x_1^{\text{down}}, x_2^{\text{down}}) = (0, 0) \qquad \text{and} \qquad (x_1^{\text{up}}, x_2^{\text{up}}) = (\pi, 0) \tag{79}$$

corresponding to the pendulum hanging completely downwards and staying completely upwards, respectively. The Jacobian of $\vec{f}$ is given by

$$J\vec{f}(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos(x_1) & -\frac{k}{m} \end{bmatrix}. \tag{80}$$

By evaluating the Jacobian at the equilibria, we get that

$$J\vec{f}(x_1^{\text{down}}, x_2^{\text{down}}) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{k}{m} \end{bmatrix} \qquad J\vec{f}(x_1^{\text{up}}, x_2^{\text{up}}) = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{k}{m} \end{bmatrix}. \tag{81}$$

One can show that the eigenvalues of $J\vec{f}(x_1^{\text{down}}, x_2^{\text{down}})$ each have negative real part, so the linearized model at $(x_1^{\text{down}}, x_2^{\text{down}})$ is stable. On the other hand, there is an eigenvalue of $J\vec{f}(x_1^{\text{up}}, x_2^{\text{up}})$ with positive real part, so the linearized model at $(x_1^{\text{up}}, x_2^{\text{up}})$ is unstable. This corresponds with our physical intuition; if we shake a pendulum which is somehow standing straight up, it will immediately fall over and hang downwards, while if we shake a pendulum which is hanging downwards, it will move around a little but will eventually return to hanging downwards.

5 Final Comments

In this note, we learned how to linearize functions, and specifically how to linearize nonlinear control models. *Linearization is what makes linear control useful*, since most physical systems are nonlinear and thus the linear control model would not apply everywhere. Thus, linearization unlocks some rudimentary nonlinear control, allowing us to use linear control methods on nonlinear models. This neatly closes the loop on the controls picture we have developed.

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