1 Complex Numbers Introduction

**Definition 1 (Complex Numbers)**
Consider an arbitrary complex number \( a \in \mathbb{C} \). We can write this complex number as \( a = x + jy \) where \( j = \sqrt{-1} \) and \( x, y \in \mathbb{R} \).

**Definition 2 (Complex Number Operations)**
Consider two complex numbers \( a, b \in \mathbb{C} \). Let \( a = x + jy \) and \( b = u + jv \) where \( x, y, u, v \in \mathbb{R} \). Addition is defined as follows:
\[
 a + b = (x + jy) + (u + jv) = (x + u) + j(y + v)
\]
and multiplication is defined as follows:
\[
 a \cdot b = (x + jy) \cdot (u + jv) = xu - yv + j(xv + uy)
\]
Note: this uses the "FOIL" technique for multiplication of real quantities.

**Definition 3 (Complex Conjugate and Magnitudes)**
Consider an arbitrary complex number \( a \in \mathbb{C} \) where we can equivalently write \( a = x + jy \) for \( x, y \in \mathbb{R} \). The complex conjugate of \( a \) is
\[
 \bar{a} = x - jy
\]
The magnitude of \( a \) is
\[
 |a| = \sqrt{a\bar{a}}
\]

2 Polar Form

We will investigate another method to write complex numbers.

**Theorem 4 (Euler’s Identity)**
Consider an arbitrary complex number \( a \in \mathbb{C} \) which we can write as \( a = x + jy \). We can equivalently write this as \( a = |a| e^{j\theta} \) where \( x = |a| \cos(\theta) \) and \( y = |a| \sin(\theta) \) (equivalently, \( \theta = \text{atan2}(y, x) \)).

\(^a\text{Here, atan2}(y, x) \text{ is a function that returns the angle from the positive x-axis to the vector from the origin to the point } (x, y). \text{ See https://en.wikipedia.org/wiki/Atan2.}\)

**Proof.** Let us write \( a = |a| e^{j\theta} \). We can show that \( x = |a| \cos(\theta) \) and \( y = |a| \sin(\theta) \), using the Taylor expansion of \( f(x) = e^x \):
\[
 a = |a| e^{j\theta}
\]
\[ \begin{align*}
= |a| \left( 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \cdots + \frac{(j\theta)^{2n}}{2n!} + \frac{(j\theta)^{2n+1}}{(2n+1)!} + \cdots \right) \\
(6)
= |a| \left( 1 + j\theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \cdots + (-1)^n \frac{\theta^{2n}}{2n!} + j(-1)^n \frac{\theta^{2n+1}}{(2n+1)!} + \cdots \right) \\
(7)
= |a| \left[ \left( 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + (-1)^n \frac{\theta^{2n}}{2n!} + \cdots \right) + j \left( \theta - \frac{\theta^3}{3!} + \cdots + (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} + \cdots \right) \right] \\
(8)
= |a| (\cos(\theta) + j\sin(\theta)) \\
= |a| \cos(\theta) + j |a| \sin(\theta) \\
(9)
\end{align*} \]

To show that \( \theta = \text{atan2}(y, x) \), consider that

\[ \frac{y}{x} = \frac{|a| \sin(\theta)}{|a| \cos(\theta)} \]

\[ \Rightarrow \theta = \arctan \frac{y}{x} \]

(11)

(12)

Instead of using regular arctan, we will use \( \text{atan2} \), two argument arctan, which protects against sign errors (i.e., to differentiate the cases when \( x \) and \( y \) are both positive or both negative) and division by zero (i.e., when \( x = 0 \)). Hence, we write

\[ \theta = \text{atan2}(y, x) \]

(13)

The plot in Figure 1 visually describes the conversion from rectangular (i.e., \( x + jy \)) form to polar form (i.e., \( |a|e^{j\theta} \))

![Figure 1: Complex number \( a \in \mathbb{C} \) depicted as a vector in the complex plane.](image)

**Corollary 5 (Complex Exponential Representations of Sine and Cosine)**

Using Theorem 4, we have that

\[ \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \]

(14)
\[ \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad (15) \]

**Proof.** Using Theorem 4 and the even/odd nature of cosine/sine respectively, we have the following direct results:

\[ e^{j\theta} = \cos(\theta) + j\sin(\theta) \quad (16) \]
\[ e^{-j\theta} = \cos(\theta) - j\sin(\theta) \quad (17) \]

From this, we have that

\[ 2\cos(\theta) = e^{j\theta} - j\sin(\theta) + e^{-j\theta} + j\sin(\theta) \quad (18) \]
\[ \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad (19) \]

and

\[ 2j\sin(\theta) = e^{j\theta} - \cos(\theta) - e^{-j\theta} + \cos(\theta) \quad (20) \]
\[ 2j\sin(\theta) = e^{j\theta} - e^{-j\theta} \quad (21) \]
\[ \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad (22) \]


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