EECS 16B Designing Information Devices and Systems II
Fall 2018 Elad Alon and Miki Lustig

Midterm 2

Name: Miki Lustig

SID: Not a student

Discussion Section and TA (Monday): Arda

Discussion Section and TA (Wednesday): None

Lab Section and TA: Sukrit

Name and SID of left neighbor: Avideh Zakor

Name and SID of right neighbor: Chunlei Liu

Instructions

- You have 120 minutes to complete this exam. Check that the exam contains 12 pages total.
- After the exam begins, write your SID in the top right corner of each page of the exam.
- Only the front pages will be scanned and graded; you can use the back pages as scratch paper.
- Do not remove any pages from the exam or unstaple the exam as this disrupts scanning. If needed, cross out any work you do not want to be graded.
- Provide explanation with every answer. Final answers with no explanation will not be given credit.

Problem	Points
1	50
2	40
3	30
4	40
5	30

Table of Unit Prefixes

Prefix	M	k	m	μ	n	p	f
Value	10^6	10^3	10^{-3}	10^{-6}	10^{-9}	10^{-12}	10^{-15}

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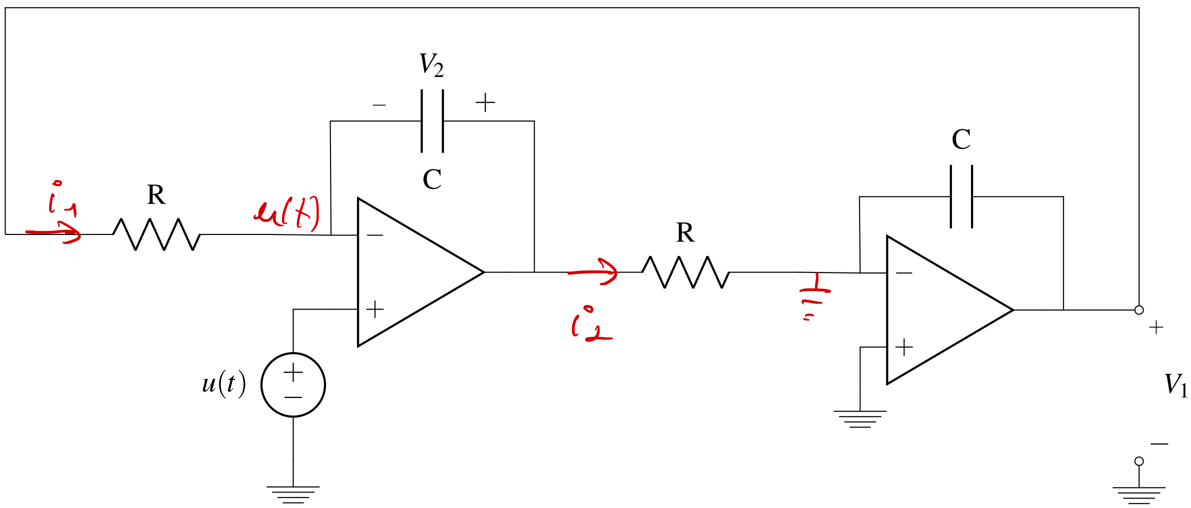
1. Circuit Controls (50 points)

Consider the following circuit with ideal op-amps:



(a) Write a state space model $\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) + Bu(t)$ with $\vec{x}(t) = \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}$.

You can assume the golden rules of op-amps apply here.

Using golden rule: $i_1 = \frac{V_1 - u(t)}{R} \Rightarrow \frac{dV_2}{dt} = \frac{u(t) - V_1}{CR}$

$i_2 = \frac{u(t) + V_2}{R} \Rightarrow \frac{dV_1}{dt} = \frac{-u(t) - V_2}{RC}$

$$\dot{\vec{x}}(t) = \begin{bmatrix} 0 & -\frac{1}{RC} \\ -\frac{1}{RC} & 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{RC} \\ +\frac{1}{RC} \end{bmatrix} u(t)$$

$$A = \begin{bmatrix} 0 & -\frac{1}{RC} \\ -\frac{1}{RC} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -\frac{1}{RC} \\ +\frac{1}{RC} \end{bmatrix}$$

(b) Consider the following continuous time system:

$$\frac{d\vec{x}(t)}{dt} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Where $0 < |a| < \infty$ and $0 < |b| < \infty$. Is the system stable?

characteristic polynomial:

$\lambda^2 - ab = 0$

case I $a>0, b>0$ or $a<0, b<0$

$\lambda^2 = |a||b|$ $\lambda_{1,2} = \pm\sqrt{|a||b|}$ not stable

$Re\{\lambda_2\} > 0$

Case II $a<0$ $b>0$ or $a>0, b<0$

$\lambda^2 = -|a||b|$ $\lambda_{1,2} = j\pm\sqrt{|a||b|}$ not stable.

$Re\{\lambda_1, \lambda_2\} = 0$

Stable / Not Stable

(c) Let $\vec{y}(t) = C\vec{x}(t)$, where $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Is the system observable?

$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix} \Rightarrow Rank\{O\} = 2$

Observable / Not Observable

(d) For the system in part (b), we design a state feedback controller $u(t) = K\vec{x}(t)$, where $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. Find the k_1 and k_2 values which will drive the system to equilibrium with eigenvalues $\lambda_1 = \lambda_2 = -1$.

Need $A + BK$ to have eigenvals of $\lambda_1 = \lambda_2 = -1$

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ k_1 & k_2 \end{pmatrix} = \begin{pmatrix} 0 & a \\ b+k_1 & k_2 \end{pmatrix}$$

characteristic polynomial:

$$-\lambda(k_2 - \lambda) - a(b+k_1) = 0$$

$$-\lambda k_2 + \lambda^2 - ab - ak_1 = 0$$

$$\lambda^2 - k_2\lambda - ab - ak_1 = 0$$

Similarly:

$$(\lambda+1)(\lambda+1) = \lambda^2 + 2\lambda + 1$$

$$-k_2 = 2 \Rightarrow k_2 = -2$$

$$+ab + ak_1 = -1$$

$$k_1 = \frac{-1-ab}{a}$$

$k_1 = -\dfrac{1+ab}{a}$ $\qquad$ $k_2 = -2$

2. System Responses (40 points)

Consider again the system:

$$\frac{d\vec{x}(t)}{dt} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \vec{x}(t)$$

Where $0 < |a| = |b| < \infty$, i.e $a = b$ or $a = -b$. For each of the following plots, state if the plot could be a possible system response for some initial state $\vec{x}(0)$. Provide a sufficient explanation to your answer. The solid line is $x_1(t)$ and the dashed line is $x_2(t)$.

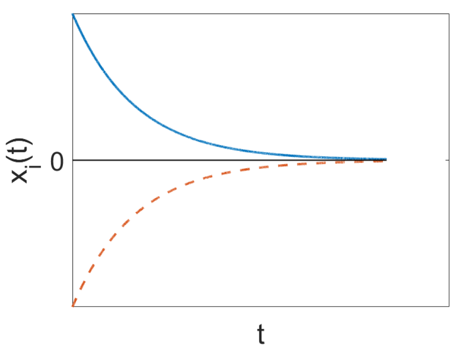
case $\underline{I}$: $a=b \Rightarrow \lambda^2 = a^2 \Rightarrow \lambda_{1,2} = \pm a$

response $\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{+at} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-at}$

case $\underline{II}$: $a=-b = \lambda^2 = -a^2 \Rightarrow \lambda_{1,2} = \pm ja$

response $\alpha \begin{bmatrix} \cos(at) \\ \sin(at) \end{bmatrix} + \beta \begin{bmatrix} \cos(at) \\ -\sin(at) \end{bmatrix}$

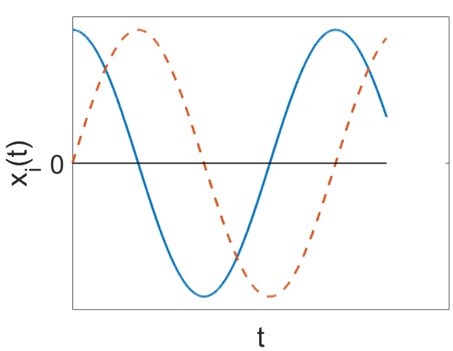
(a)



Possible? Yes / No
Explanation:

initial value is eigenvector with decaying exponential for case $\underline{I}$

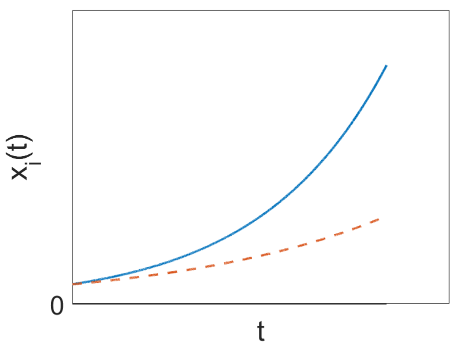
(b)



Possible? Yes / No
Explanation:

oscilatory for case $\underline{II}$

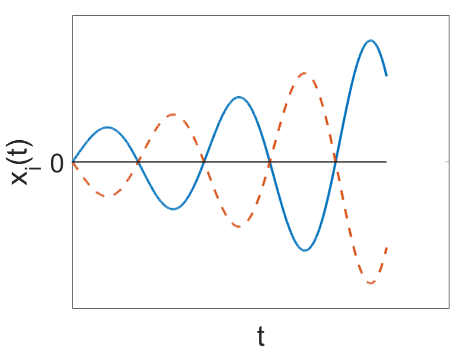
(c)



Possible? Yes / No
Explanation:

there is only explotion with e^{at} not two rates.

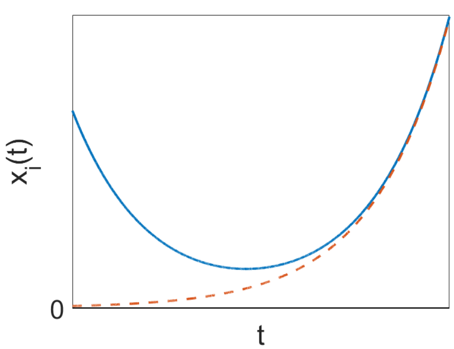
(d)



Possible? Yes / No
Explanation:

There are no complex eigenvals just pure real or imaginary

(e)



Possible? Yes / No
Explanation:

since the results are not oscilatory the most likely form of solution is:

$$\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{+at} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-at}$$

while in general decay and explode could be a solution, the above curve is impossible.

For x_1 to drop and then rise, $\beta \gg \alpha$

for x_2 to start at zero and explode $\beta = \alpha$

so the answer is no.

Thanks for Hermish Mehta for pointing out the problem in the original solution.

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3. Discrete Time System (30 points)

Consider the discrete-time system

$$x(t+1) = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Where $-\infty < a < \infty, -\infty < b < \infty$

(a) Under what conditions on a, b is the system stable?

Case I: $a>0, b>0$ $\quad a<0$ $b<0$ $\quad \lambda_{1,2} = \pm\sqrt{|a||b|}$

Case II: $a<0, b>0$ $\quad a>0$ $b<0$ $\quad \lambda_{1,2} = \pm j\sqrt{|a||b|}$

stable if $\sqrt{|a||b|} < 1$

$|a||b| < 1$

$$\boxed{|a||b| < 1}$$

(b) Determine the inputs of an open-loop controller that will take the system from $\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\vec{x}(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

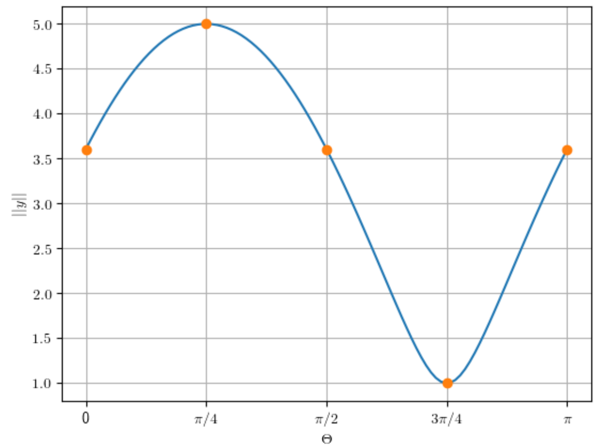
$R = [AB \; B] = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \cdot \vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$u(0) = \frac{1}{a}$	$u(1) = 1$

4. SVD (40 points)

(a) Let $A \in \mathbb{R}^{2\times 2}$ and $\vec{x} = \begin{bmatrix}\sin(\theta)\\ \cos(\theta)\end{bmatrix}$, $||\vec{x}|| = 1$. Now let $\vec{y} = A\vec{x}$. Below is the plot of $||\vec{y}||$ vs θ.



What can we learn of the SVD of A? In the space provided below, complete the matrices that can be determined with the above information, and explain what's missing.

We know that $\sigma_2 \le ||Ax|| \le \sigma_1$

so $\sigma_1 = 5$ $\sigma_2 = 1$

These occure for $x = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} = v_1$ and $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix} = v_2$

since we obsere $||y||$ ~~u~~ u can be arbitrarily rotated, so we don't know u

$$U = \begin{bmatrix} ? \end{bmatrix} \quad S = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \quad V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

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(b) Let $A \in \mathbb{R}^{N \times N}, B \in \mathbb{R}^{N \times N}$ be full rank matrices and let $\vec{x} \in \mathbb{R}^N$ have $||\vec{x}|| = 1$. We compute $\vec{y} = A \cdot B \cdot \vec{x}$. Find the upper bound for $||\vec{y}||$ in terms of the singular values of A and B. Explain your answer

$$||Bx|| \leq \sigma_{max}\{B\}$$

$$\left|\left|A \frac{\tilde{x}}{||\tilde{x}||}\right|\right| \leq \sigma_{max}\{A\}$$

If $x = V_1\{B\}$ and $Bx = V_1\{A\}$

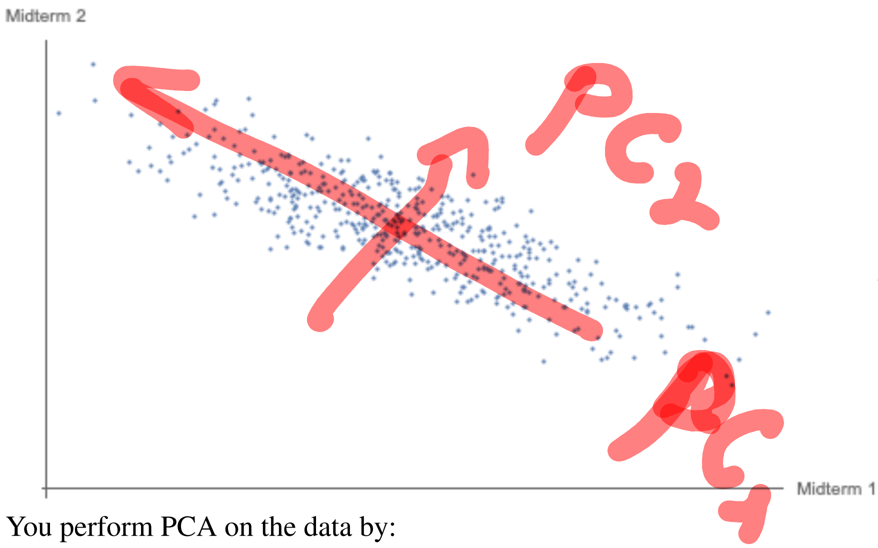
Then the output is maximal

with $||ABx|| = \sigma_{max}\{A\} \cdot \sigma_{max}\{B\}$

$||y|| \leq \sigma_{max}\{A\}\sigma_{max}\{B\}$

5. Data Science (30 pts)

After midterm 2, we conducted a survey in which we asked students to rate the difficulty of midterms 1 and 2 on a continuous scale from 0 to 10. The results of the scatter plots are below.



$$A = \begin{array}{c} \begin{array}{cc} \text{mid}_1 & \text{mid}_2 \end{array} \\ \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \\ a_n & b_n \end{bmatrix} \end{array}$$

You perform PCA on the data by:

(1) Subtract the mean of each column and store the demeaned data in $\tilde{A}$

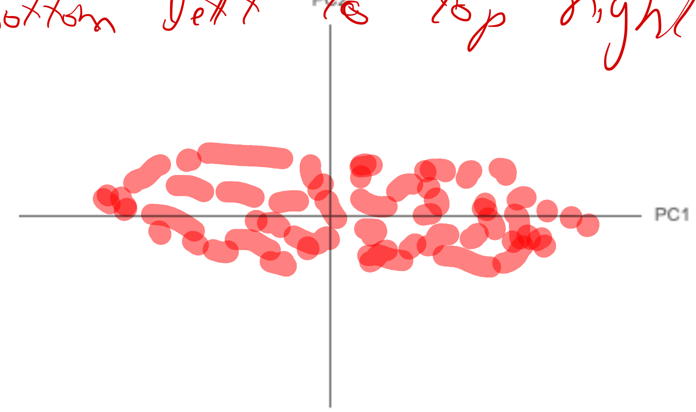
(2) Compute the SVD: $\tilde{A} = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T$

(3) $\tilde{A}^T u_1 = \begin{bmatrix} -41.7 \\ 46.9 \end{bmatrix}, \quad \tilde{A}^T u_2 = \begin{bmatrix} 8.6 \\ 7.6 \end{bmatrix}$

(a) Draw a scatter plot of the projected $\tilde{A}v_1, \tilde{A}v_2$ points on the PCA basis. Explain your answer.

Because of 3, mid_1 has negative inner product with PC_1 so PC_1 points right $\rightarrow$ left

PC_2 Points bottom left to top right

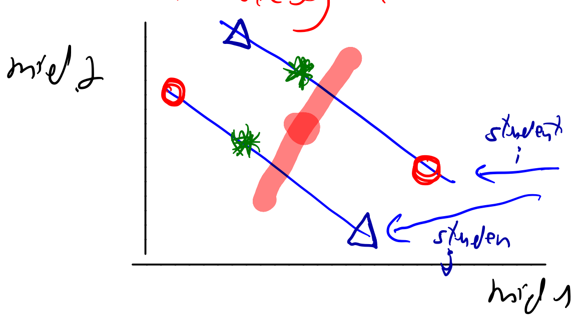


(b) Let $[a_i, b_i]$ be the rating of the ith student, and $\vec{v}_2$ be the second principal component vector. What would it mean for $\begin{bmatrix} a_i & b_i \end{bmatrix} \vec{v}_2 > \begin{bmatrix} a_j & b_j \end{bmatrix} \vec{v}_2$? Circle the answer within each set of slashes. Explain your answer.

u_2 looks to have positive inner product with both $\vec{a}$ and $\vec{b}$ so $\vec{v}_2$ should also point ↗ rather than ↙

This direction indicates more difficulty on both exams.

Unfortunately this is not enough information...



all points similar projection on $\vec{v}_2$ lie on this line.

○ case for student i finding mid 1 more difficult but mid 2 easier than student j

△ case for student i finding mid 2 more difficult, but mid 1 easier than student j

✱ case for student i finding both midterms harder than student j

so, all answers could be right except student j finding both midterms harder than student i

all except
~~Student~~ i / j found midterm(s) 1 / 2 / 1&2

more / less difficult than student i / j found midterm(s) 1 / 2 / 1&2.