Midterm 2

ⓘ This is a preview of the published version of the quiz

Started: Apr 15 at 10:38pm

Quiz Instructions

This midterm will be open notes, open Internet, and open-calculator; but you may not consult another person while taking the exam.

Question 1

1 pts

A dynamical system model for an epidemic with total population $N = S + I + R$, where S is the number of susceptible individuals, I is the number of infected, and R is the number of recovered, is modeled by

$$\frac{d}{dt}S = -\beta\frac{IS}{N}$$
$$\frac{d}{dt}I = \beta\frac{IS}{N} - \gamma I$$
$$\frac{d}{dt}R = \gamma I$$

Here, we use real numbers since integer granularity is not required. Consider the situation before the onset of the epidemic, with $S = N$, $I = 0$, and $R = 0$. The linearized state-space model is given by

$$\frac{d}{dt}\begin{bmatrix}\tilde{s}\\ \tilde{i}\\ \tilde{r}\end{bmatrix} = A\begin{bmatrix}\tilde{s}\\ \tilde{i}\\ \tilde{r}\end{bmatrix},$$

where the lower case variables with tildes are the linearized variables for the model. Then, the matrix A is given by:

- ○ $A = \begin{bmatrix}-\beta & -\beta & 0\\ \beta & \beta-\gamma & 0\\ 0 & \gamma & 0\end{bmatrix}$

- ○

$$A = \begin{bmatrix} 0 & -\beta & 0 \\ 0 & \gamma - \beta & 0 \\ 0 & \gamma & 0 \end{bmatrix}$$

- $$A = \begin{bmatrix} 0 & -\beta & 0 \\ 0 & \beta - \gamma & 0 \\ 0 & \gamma & 0 \end{bmatrix}$$
- $$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
- $$A = \begin{bmatrix} -\beta & -\beta & 0 \\ 0 & \beta - \gamma & 0 \\ 0 & \gamma & 0 \end{bmatrix}$$

Question 2

1 pts

A system $\frac{d}{dt}\vec{x} = A\vec{x} + B\vec{u}$ has controllability matrix $\mathcal{C} = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$.

Suppose that $\vec{z} = T\vec{x}$, where T is an invertible matrix. What is the controllability matrix for the system resulting from this change of coordinates?

- $T\mathcal{C}$
- $T\mathcal{C}T^{-1}$
- $\mathcal{C}T^{-1}$
- $\mathcal{C}$
- $T^{-1}\mathcal{C}$

Question 3

1 pts

Given the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

Which of the following are true statements about the Singular Value Decomposition (SVD) of A?

1. All eigenvalues λ_i of $AA^\top$ are identical to each other.
2. Non zero singular values are $\sigma_1 = 3, \sigma_2 = 2, \sigma_3 = 1$.
3. Removing the last row of A doesn't change the non-zero singular values.

- 1 and 2 only.
- 2 and 3 only.
- 1 and 3 only.
- 1 only.
- 1, 2, and 3.

Question 4

1 pts

Which of the following statements about the Singular Value Decomposition (SVD) is true when written in the form $A = \sigma_1 \vec{u_1} \vec{v_1}^\top + \sigma_2 \vec{u_2} \vec{v_2}^\top + \dots$? Assume that all σ_i, the singular values, are non-zero.

- $\{\vec{u_1}, \vec{u_2}, \dots\}$ is an orthonormal basis for the column space of A.
- The singular values, σ_i, are real numbers of arbitrary sign.
- The SVD separates a rank r matrix A into a sum of $r - 1$ rank 1 matrices.
- The SVD of a matrix A is unique.
- None of the others.

Question 5

1 pts

The dynamics of an epidemic, with a fixed population N are sometimes modeled with a state-space model of the form:

$$\frac{d}{dt}S = -\beta\frac{IS}{N}$$
$$\frac{d}{dt}I = \beta\frac{IS}{N} - \gamma I$$
$$\frac{d}{dt}R = \gamma I$$

where S is the number of susceptible individuals, I is the number of infected individuals , R is the number of recovered individuals, and $N = S + I + R$ is the total population. Although numbers of individuals are integer valued, we use real numbers in this exercise since integer granularity is not needed. Positive constants β and γ parametrize the epidemic dynamics.

How many equilibrium points does the epidemic dynamics of the model above have?

- 3
- Infinitely many
- 1
- 2
- 0

Question 6

1 pts

When the system $\frac{d}{dt}\vec{x} = A\vec{x}$ is discretized at a certain sampling period, the resulting discrete-time state space model is $\vec{x}_d(t+1) = A_d\vec{x}_d(t)$.

What is the state space model when $\frac{d}{dt}\vec{x} = 2A\vec{x}$ is discretized at the same sampling period?

- $\vec{x}_d(t+1) = 2A_d\vec{x}_d(t)$
- $\vec{x}_d(t+1) = A_d^2\vec{x}_d(t)$
- $\vec{x}_d(t+1) = (A_d + 2I)\vec{x}_d(t)$
- $\vec{x}_d(t+1) = (A_d^2/2 + I)\vec{x}_d(t)$
- Not enough information to determine

Question 7

1 pts

Suppose the following linear dynamical system is controllable:

$$\frac{d}{dt}\vec{x} = \mathbf{A}\vec{x} + \vec{b}_1 u$$

Which additional conditions are necessary for the following system to be controllable?

$$\frac{d}{dt}\vec{x} = \mathbf{A}\vec{x} + \mathbf{B}\vec{u}$$

where $\mathbf{B} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}$.

- The system $\frac{d}{dt}\vec{x} = \mathbf{A}\vec{x} + \vec{b}_2 u$ must also be controllable.
- The system cannot be controllable under any conditions.
- None, the system is already controllable.
- $\mathbf{A}$ and $\mathbf{B}$ have orthogonal columns.
- $\vec{b}_1$ and $\vec{b}_2$ must be orthogonal.

Question 8

1 pts

Suppose we have a linear dynamical system $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$

where $\vec{x}(t) \in \mathbb{R}^n$ and $\vec{u}(t) \in \mathbb{R}^m$.

Which of the following are necessarily true:

I. $\vec{x} = 0$ is an equilibrium point for $\vec{u} = 0$.

II. For any given input $\vec{u}$, there must exist a unique equilibrium point $\vec{x}^*$.

III. Suppose $(\vec{x}^*, \vec{u}^*)$is an equilibrium point, $\vec{x}(0) = \vec{x}^*$, and $\vec{u}(t) = \vec{u}^*$ for all $t \geq 0$. Then $\vec{x}(t)$is constant for $t \geq 0$.

IV. If A is invertible, there exists an input for which there are no equilibrium points.

V. If $\vec{x}_1^*$and $\vec{x}_2^*$are equilibrium points for $\vec{u} = 0$, $\vec{x}_1^* + \vec{x}_2^*$is also an equilibrium point.

- I only.
- II, III, IV
- I, III, V
- I, II, III, IV
- I, II, III, IV, V

Question 9

1 pts

Consider the discrete time system

$$\vec{x}(k+1) = A\vec{x}(k) + \vec{b}u(k)$$

with $\vec{x}(\cdot) \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times3}$, and $\vec{b} \in \mathbb{R}^3$.

Suppose that the system is controllable from the origin $\vec{x}(0) = 0$ in 10 steps. That is, one can design a control sequence $\{u(0), u(1), \ldots, u(9)\}$ to reach any target state $\vec{x}^* = \vec{x}(10)$ in 10 steps. Which of the following is true?

- For any target state $\vec{x}^*$, one can find an initial condition $\vec{x}(0)$and a two step input sequence $\{u(0), u(1)\}$ to reach $\vec{x}^*$.
- None of the other answers is correct.

- Any state $\vec{x}^*$ can be also be reached with a shorter input sequence $\{u(0), u(1)\}$ in two steps.
- The state $\vec{x}^*$ cannot be reached from the origin in 9 steps with any possible sequence $\{u(0), u(1), \ldots, u(8)\}$.
- The input sequence $\{u(0), u(1), \ldots, u(9)\}$ to reach $\vec{x}^*$ is unique.

Question 10

1 pts

How many non-zero singular values does the following matrix A have?

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \\ 4 & 4 & 8 \\ 5 & 1 & 2 \end{bmatrix}$$

- 1
- 5
- 2
- 4
- 3

Question 11

1 pts

Suppose we have the relation $\vec{y} = D\vec{p} + \vec{e}$, as seen from lecture. In order to determine $\vec{\hat{p}}$, the least squares estimate, which of the following assumptions were made?

- D is diagonal.
- $D^\top D$ is invertible.

- $\vec{e}$ is orthogonal to $\vec{y}$.
- None of the others assumptions.
- $D^\top$ is invertible.

Question 12

1 pts

Consider the scalar system $x(t+1) = bu(t) + e(t)$, where, b is the only unknown parameter and $e(t)$ is a disturbance term. Suppose, we apply the input, $u(0) = u(1) = u(2) = u(3) = 1$and observe the resulting state trajectory to obtain a least-squares estimate $\hat{b}$ for b. Which of the following state trajectories would result in the estimate $\hat{b}$ = 1?

- $x(1) = 1.1, x(2) = 0.9, x(3) = 1.2, x(4) = 1$
- $x(1) = 0.1, x(2) = 0.9, x(3) = 1.7, x(4) = 1.2$
- $x(1) = 0.1, x(2) = 1.9, x(3) = 1, x(4) = 0.9$
- $x(1) = 1.2, x(2) = 0.9, x(3) = 0.6, x(4) = 1.0$
- $x(1) = 0.1, x(2) = 1.1, x(3) = 1.9, x(4) = 0.9$

Question 13

1 pts

Which of the following are true about the Singular Value Decomposition (SVD)?

1. If a square matrix Q is orthonormal ($QQ^\top = I$), then its singular values are all 1.
2. A matrix with rank r will have exactly r singular values greater than 0.
3. Every real matrix has an SVD.

- 1 only.
- 2 and 3 only.

- 1 and 2 only.
- 1, 2, and 3.
- 1 and 3 only.

Question 14

1 pts

Consider a linear system, $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$, where $\vec{x}(t) \in \mathbb{R}^n$ and $\vec{u}(t) \in \mathbb{R}^m$.

Which of the the following conditions can, on its own, determine whether the system is **controllable or not**?

I.	$m < n$
II.	$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
III.	$m = n$ and B is invertible
IV.	$AB = 0$ and $m < n$
V	$\textbf{rank}(A) = n$

- II, III, and IV only
- I, II, III, IV, and V
- I, III, and V only
- I, II, III, and IV only
- II and III only

Question 15

1 pts

Consider the discrete time dynamical system

$$y(k+1) = b_1 u(k) + b_2 u(k-1) + e(k),$$

where $e(k)$ accounts for additive noise, and we get to measure the $y(\cdot)$ and the $u(\cdot)$ data sequences exactly. We set up an estimation scheme to estimate the unknown real parameters b_1, and b_2:

$$\begin{bmatrix} u(1) & u(0) \\ u(2) & u(1) \\ \vdots & \vdots \\ u(N) & u(N-1) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} y(2) \\ y(3) \\ \vdots \\ y(N+1) \end{bmatrix}.$$

Suppose that $u(k) = \lambda^k$. For this input, what is the minimum number of steps, i.e. samples of $y(\cdot)$, needed to uniquely estimate the parameters b_1 and b_2?

- 2
- 1
- 4
- Cannot be uniquely estimated, no matter how many samples
- 3

Question 16

1 pts

Consider the following dynamical system:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t)x_2(t) + u(t)x_1^2(t) \\ \cos\left(\frac{\pi}{2}x_1(t)\right) \end{bmatrix}$$

For $u(t) = 1$, consider the following equilibrium point $\vec{x}^* = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Let $\vec{\tilde{x}}(t) = \vec{x}(t) - \vec{x}^*$ and $\tilde{u}(t) = u(t) - 1$. We wish to write a system as

$$\frac{d}{dt}\vec{\tilde{x}}(t) = A\vec{\tilde{x}}(t) + B\tilde{u}(t)$$

Which of the following is a correct linearization:

-

$A = \begin{bmatrix} 1 & 1 \\ -\frac{\pi}{2} & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- $A = \begin{bmatrix} x_2(t) & x_1(t) \\ -\frac{\pi}{2}\sin\left(\frac{\pi}{2}x_1(t)\right) & 0 \end{bmatrix}, B = \begin{bmatrix} x_1^2(t) \\ 0 \end{bmatrix}$
- $A = \begin{bmatrix} x_2(t) + 2u(t)x_1(t) & x_1(t) \\ -\frac{\pi}{2}\sin\left(\frac{\pi}{2}x_1(t)\right) & 0 \end{bmatrix}, B = \begin{bmatrix} x_1^2(t) \\ 0 \end{bmatrix}$
- $A = \begin{bmatrix} 1 & -\frac{\pi}{2} \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \end{bmatrix}$
- $A = \begin{bmatrix} 1 & 1 \\ 0 & \frac{\pi}{2} \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Question 17

1 pts

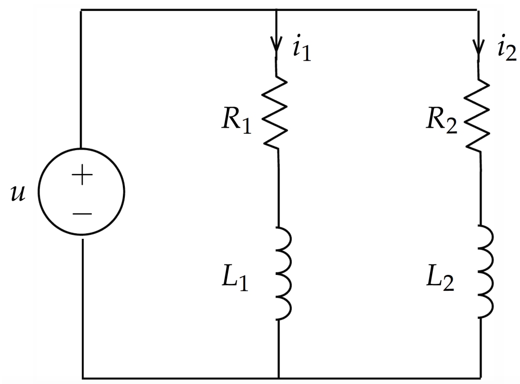
Which of the following is a valid SVD for $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$?

- $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \sigma_1 = 1, \sigma_2 = 1$
- $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \sigma_1 = 1, \sigma_2 = 1$
- $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \sigma_1 = 1, \sigma_2 = -1$
- $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \sigma_1 = 0.5, \sigma_2 = 0.5$
- $\vec{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \sigma_1 = 1, \sigma_2 = 1$

Question 18

1 pts

Consider the circuit below, where $u(t)$ is the input and $i_1(t)$and $i_2(t)$ are the state variables:



Suppose, $R_1 = 1$ mΩ, $L_1 = 1$ mH , $L_2 = 2$ mH. For which value of R_2 is this system uncontrollable?

- $\mathrm{R_2} = 1\ \mathrm{m\Omega}$
- $\mathrm{R_2} = 0\ \Omega$
- $\mathrm{R_2} = 2\ \mathrm{m\Omega}$
- None. It is controllable for all values of R_2.
- $\mathrm{R_2} = 0.5\ \mathrm{m\Omega}$

Question 19

1 pts

Let A be an $m \times n$ real matrix with SVD in standard outer product form

$A = \sigma_1 \vec{u}_1 \vec{v}_1^\top + \sigma_2 \vec{u}_2 \vec{v}_2^\top + \sigma_3 \vec{u}_3 \vec{v}_3^\top$ with $\sigma_1 \geq \sigma_2 \geq \sigma_3 > 0$.

Which of the following is NOT true:

- $A^\top A\, \vec{v}_2 = \sigma_2^2 \vec{v}_2$
- $n \geq 3$

- $\mathrm{rank}(A^\top) = 3$
- $\vec{v}_1 \vec{v}_1^\top = 1$
- $\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}^\top \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Question 20

1 pts

Consider the system:

$$\frac{dx(t)}{dt} = (a - by(t))x(t)$$
$$\frac{dy(t)}{dt} = (cx(t) - d)y(t)$$

where, $x(t)$ and $y(t)$ are non-negative state variables and a, b, c, and d are positive constants. Professor Arcak linearized this model around one of its equilibrium points (he won't tell you which) and found that the resulting matrix A has complex eigenvalues. What are these eigenvalues?

- $\lambda_{1,2} = \pm j\sqrt{ad}$
- $\lambda_{1,2} = -bd/c \pm jac/b$
- $\lambda_{1,2} = a \pm jd\sqrt{b/c}$
- $\lambda_{1,2} = -d \pm ja$
- $\lambda_{1,2} = -d \pm ja\sqrt{c/b}$

Question 21

1 pts

A linear dynamical system is given below:

$$\frac{d}{dt}\vec{x} = \mathbf{A}\vec{x} + \mathbf{B}\vec{u}$$

The input $\vec{u}$ is a constant. What property of the matrix $\mathbf{A}$is required so that the system has exactly two distinct equilibrium points?

- Always possible
- Not possible
- $\mathbf{B}\vec{u}$ is in the column space of $\mathbf{A}$
- The system is controllable
- $\mathbf{A}$ is not invertible

Question 22

1 pts

An invertible $n \times n$ matrix A has n distinct non-zero singular values. How many singular value decompositions $A = U\Sigma V^\top$ does A have?

- 2^{n-1}
- n^2
- $n!$
- 2^n
- Not enough information to determine

Question 23

1 pts

Which of the following could be a non-zero singular value for matrix B below?

$$B = \begin{bmatrix} 1 & 5 & 1 & 1 & 2 \\ 2 & 7 & 2 & 9 & 4 \\ 3 & 3 & 3 & 4 & 6 \end{bmatrix}$$

- 1.01+2.14j
- -1.05
- 1.01-2.14j
- -100
- 4.04

Question 24

1 pts

A discrete-time system is modeled by the following equation:
$x(t+1) = ax(t) + bu(t) + e(t)$, where $e(t)$ is the system disturbance. The inputs and outputs at different time steps are :
$x(0) = 1,\ x(1) = 2,\ x(2) = 1,\ x(3) = -2,\ u(0) = 1,\ u(1) = 0, u(2) = 1.$
What are the least-squares estimates of the parameters a and b?

- $a = \frac{1}{2}$ and $b = 1$
- $a = \frac{1}{2}$ and $b = -\frac{1}{2}$
- $a = 1$ and $b = -\frac{1}{2}$
- $a = 1$ and $b = 1$
- $a = 1$ and $b = -1$

Question 25

1 pts

Consider the continuous-time system

$$\frac{dx_1(t)}{dt} = x_2(t)$$
$$\frac{dx_2(t)}{dt} = u(t)$$

where $u(t)$ is the input. Professor Sanders discretized this model with a sampling period T and obtained,

$$\vec{x_d}(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x_d}(k) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u_d(k).$$

What is the sampling period, T Professor Sanders used?

- $T = 0.5$
- $T = 1$
- $\mathrm{T} = 1/\sqrt{2}$
- $T = 0.1$
- $T = 0.2$

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