EECS 16B Summer 2020 Final (Form 1)

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. . . .

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> *Should I commit academic misconduct during this exam, let me receive a failing grade in EECS 16B or dismissal from the University.*

. . . .

1 Abstract singular values

Let $M \in \mathbb{C}^{N\times N}$ be a matrix that satisfies $M^N = I$ and $M^*M = MM^* = I$. Let $\omega = e^{2\pi j/N}$.

a) Show that the eigenvalues of M are powers of ω.[1]

Solution

Let $\lambda \in \mathbb{C}$ be an eigenvalue of M with eigenvector v.

$$\begin{aligned} Mv &= \lambda v \\ \implies M^N v &= \lambda^N v \\ \implies \quad v &= \lambda^N v \\ \implies \quad 0 &= (\lambda^N - 1)v \\ \implies \quad 0 &= \lambda^N - 1 \end{aligned}$$

The Nth roots of unit are the powers of ω.

b) Next we will compute the SVD of $A = M - M^2$. Show that $A^*A = 2I - M - M^{N-1}$.

Solution

$$\begin{aligned} A^*A &= \left(M - M^2\right)^* \left(M - M^2\right) \\ &= \left(M^* - \left(M^2\right)^*\right)\left(M - M^2\right) \\ &= \left(M^{-1} - M^{-2}\right)\left(M - M^2\right) \\ &= I - M - M^{-1} + I \\ &= 2I - M - M^{N-1} \end{aligned}$$

c) Show that the eigenvalues of A^*A are $2\left(1 - \cos(2\pi n/N)\right)$, $n = 0, 1, \ldots, N-1$. Conclude that they are nonnegative.

Solution

As $A^*A = 2I - M - M^{N-1}$, the eigenvalues of A^*A are $2 - \mu - \mu^{N-1}$ for all eigenvalues μ of M:

$$\begin{aligned} &\left\{2 - \mu - \mu^{N-1} \,\middle|\, \mu \text{ is an eigenvalue of } M\right\} \\ =& \left\{2 - \omega^n - \omega^{n(N-1)} \,\middle|\, n = 0, 1, \ldots, N-1\right\} \\ =& \left\{2 - \omega^n - \omega^{-n} \,\middle|\, n = 0, 1, \ldots, N-1\right\} \\ =& \left\{2 - 2\,\mathrm{Re}\{\omega\} \,\middle|\, n = 0, 1, \ldots, N-1\right\} \\ =& \left\{2 - 2\cos(2\pi n/N) \,\middle|\, n = 0, 1, \ldots, N-1\right\} \end{aligned}$$

[1] A typo in the original exam was corrected.

All cosines lie between -1 and 1 (west- and eastmost ends of the unit circle), so all of these eigenvalues are nonnegative.

d) What is a minimal singular value of A? What does this say about the rank of A?

Solution

For $n = 0$ above there is an eigenvalue 0 of A^*A. That means a singular value of A is $\sqrt{0} = 0$, and A has rank less than N.

e) If N is even, what is a maximal singular value of A?

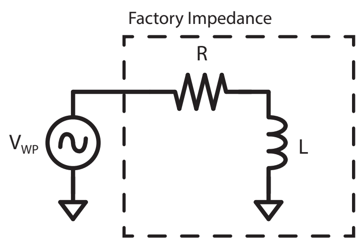
Solution

When $n = N/2$, A^*A has an eigenvalue of 4. The maximal singular value of A is is $\sqrt{4} = 2$.

2 Factory Power

A factory has several very large motors with large inductance. The factory has noticed that their power bill is high and they would like to lower it.

a) The factory's power draw is more efficient if their load impedance is purely real at f_{wp}. Let us model the factory's impedance load as shown below. The wall power has frequency $f_{wp} = 60Hz$, $L = 20\,\text{H}$, and $R = 200\,\Omega$. Add a useful device, either in series or in parallel, to their impedance load and solve for it's optimum value.



Solution

Add a series capacitor anywhere to zero imaginary impedance at f_{wp} We can ignore R and look only at the imaginary part to solve for C.

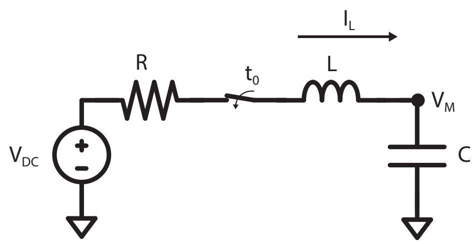
$$\omega_{wp}^2 = \frac{1}{LC}$$

$$(2\pi 60)^2 = \frac{1}{20C_F}$$

$$C_F = \frac{1}{20(2\pi 60)^2}$$

$$C_F = 0.35\mu\,\text{F}$$

b) The factory has a machine, with a DC power source, that flickers when it turns on.



Write the differential equation for motor voltage $V_M(t)$ in terms of L, C, and R that describes the given circuit in form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} V_{\mathrm{M}} \\ I_{\mathrm{L}} \end{bmatrix} = A\begin{bmatrix} V_{\mathrm{M}} \\ I_{\mathrm{L}} \end{bmatrix} + b$$

Solution

From KVL and KCL we can relate currents and voltages in our schematic.

$$I_L = I_C = I_R$$
$$V_{IN} = V_R + V_L + V_M$$

We can also write our voltage and current relationship for R_G, R_{TA}, L_{TA}, and C_{TA}.

$$\frac{\mathrm{d}}{\mathrm{d}t} V_M C = I_L$$
$$\frac{\mathrm{d}}{\mathrm{d}t} I_L L = V_L$$
$$V = I_L R$$

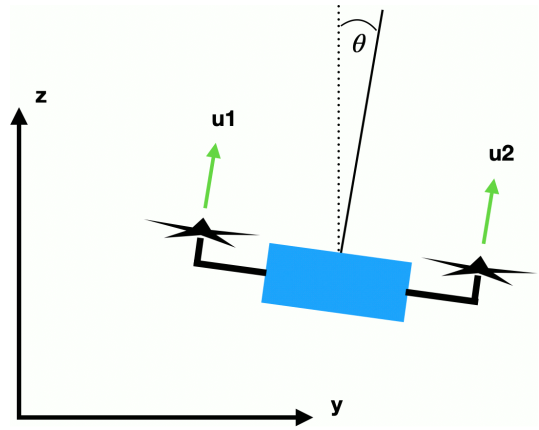
Combining all of the above we get equations for $\frac{\mathrm{d}}{\mathrm{d}t} V_M$ and $\frac{\mathrm{d}}{\mathrm{d}t} I_L$

$$\frac{\mathrm{d}}{\mathrm{d}t} V_M C = I_L$$
$$\frac{\mathrm{d}}{\mathrm{d}t} I_L L = V_{IN} - I_L R - V_M$$

Which we rearrange to write in a form that matches our vector differential equation.

$$\frac{\mathrm{d}}{\mathrm{d}t} V_M = \frac{I_L}{C}$$
$$\frac{\mathrm{d}}{\mathrm{d}t} I_L = \frac{V_{IN}}{C} - \frac{I_L R}{C} - \frac{V_M}{C}$$
$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}$$
$$b = \begin{bmatrix} 0 \\ \frac{V_I}{L} \end{bmatrix}$$

3 Controlling a Quadrotor to Hover



In this problem you will design a controller which will make a planar quadrotor hover. The quadrotor we will consider is defined by the following state space model:

$$\begin{bmatrix} \dot{y}(t) \\ \dot{v_y}(t) \\ \dot{\theta}(t) \\ \dot{\omega}(t) \\ \dot{z}(t) \\ \dot{v_z}(t) \end{bmatrix} = \begin{bmatrix} v_y(t) \\ \frac{sin(\theta(t))}{m}(u_1(t) + u_2(t)) \\ \omega(t) \\ \alpha(u_1(t) - u_2(t)) \\ v_z(t) \\ \frac{cos(\theta(t))}{m}(u_1(t) + u_2(t)) - g \end{bmatrix}$$

Here $y(t)$ denotes lateral position, $z(t)$ the altitude, $v_y(t)$ and $v_z(t)$ the corresponding linear velocities, $\theta(t)$ the roll angle, and $\omega(t)$ the angular velocity. The parameters α and m are positive, real constants. The controls $u_1(t)$ and $u_2(t)$ are the thrusts generated by the left and right propellors.

The thrust of each propellor can be positive or negative.

Define the vectors

$$x(t) := \begin{bmatrix} y(t) \\ v_y(t) \\ \theta(t) \\ \omega(t) \\ z(t) \\ v_z(t) \end{bmatrix}, \quad u(t) := \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

a) An equilibrium point for this system is given by

$$x^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ h \\ 0 \end{bmatrix}, \quad u^* = \begin{bmatrix} \frac{mg}{2} \\ \frac{mg}{2} \end{bmatrix}$$

Here $h > 0$ is a specified altitude.

Do there exist any other equilibrium points for this system which satisfy $y^* = 0$ and $z^* = h$? If so, what are they? If not, explain why not.

Solution

Yes there exist infinite such equilibria (or two if θ is modulated by 2π.) The equilibra are given by

$$\begin{bmatrix} y^* \\ v_y^* \\ \theta^* \\ \omega^* \\ z^* \\ v_z^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k\pi \\ 0 \\ h \\ 0 \end{bmatrix}, \quad \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix} = \begin{bmatrix} (-1^k)\frac{mg}{2} \\ (-1^k)\frac{mg}{2} \end{bmatrix}, \quad k \in \mathbb{Z}$$

b) Consider a linearization of this system, formed by taking the first-order taylor approximation of the system about the equililbrium point given in part (a). This linearized system is given by

$$\dot{\delta x}(t) = A\delta x(t) + B\delta u(t)$$

where $\delta x(t) = (x(t) - x^*)$, and $\delta u(t) = (u(t) - u^*)$. The matrices A and B are given by

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \beta_3 & -\beta_3 \\ 0 & 0 \\ \beta_4 & \beta_4 \end{bmatrix}$$

Find the parameters $\beta_1, \beta_2, \beta_3, \beta_4$.

Solution

$$\beta_1 = \left.\frac{\partial(\frac{sin\theta}{m}(u_1 + u_2))}{\partial\theta}\right|_{\theta=\theta^*, u_1=u_1^*, u_2=u_2^*} = \frac{cos\theta^*}{m}(u_1^* + u_2^*) = g$$

$$\beta_2 = \left.\frac{\partial v_z}{\partial v_z}\right|_{v_z=v_z^*} = 1$$

$$\beta_3 = \left.\frac{\partial\alpha(u_1 - u_2)}{\partial u_1}\right|_{u_1=u_1^*, u_2=u_2^*} = \alpha$$

$$\beta_4 = \left.\frac{\partial(\frac{cos\theta}{m}(u_1 + u_2) - g)}{\partial u_1}\right|_{\theta=\theta^*, u_1=u_1^*, u_2=u_2^*} = \frac{cos\theta^*}{m} = \frac{1}{m}.$$

c) Is the linearized system found in part (b) stable? Explain your answer. Hint: notice that A is an upper-triangular matrix.

Solution

The system is marginally stable, since all 6 eigenvalues are 0. This is clearly seen by the fact that the eigenvalues of an upper-triangular matrix appear along the main diagonal of the matrix. Because this system is marginally stable, it is unstable.

d) Does the matrix B have full column-rank? Explain your answer. Here you can use the fact $\beta_3 \neq 0$ and $\beta_4 \neq 0$.

Solution

Yes. Let $b_i(k)$ denote the kth element of the ith column of B. The scalar γ which solves $\gamma b_1(4) = b_2(4)$ is $\gamma = -1$. But $\gamma b_1(6) \neq b_2(6)$. This implies b_1 and b_2 are linearly independent, and B is full column-rank.

e) Consider the matrix $C = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$. Is C full row-rank? What does this imply about the ability or inability to choose arbitrary closed-loop eigenvalues for this system through use of feedback control? Explain your answer.

Solution

$$C = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \left[\begin{array}{cc|cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 & \beta_1\beta_3 & -\beta_1\beta_3 \\ 0 & 0 & 0 & 0 & \beta_1\beta_3 & -\beta_1\beta_3 & 0 & 0 \\ 0 & 0 & \beta_3 & -\beta_3 & 0 & 0 & 0 & 0 \\ \beta_3 & -\beta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_2\beta_4 & \beta_2\beta_4 & 0 & 0 & 0 & 0 \\ \beta_4 & \beta_4 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Each row is zero except in exactly one of the matrices $\{B, AB, A^2B, A^3B\}$. Each of these matrices is full rank, implying therefore that each row of C is linearly independent. Therefore C is full row-rank. Because C is full row-rank, the system (A, B) is controllable, and therefore we can design a control law $\delta u(t) := -K\delta x(t)$ such that the closed loop eigenvalues of the resultant system can be arbitrariliy chosen.

f) Define a control law for this system of the form $\delta u(t) := -K\delta x(t)$, where K is defined as

$$K := \begin{bmatrix} 0 & 0 & 0 & \frac{k_1}{2} & 0 & \frac{k_2}{2} \\ 0 & 0 & 0 & -\frac{k_1}{2} & 0 & \frac{k_2}{2} \end{bmatrix}.$$

Find the constants k_1 and k_2 in terms of the parameters β_3 and β_4 so that two of the eigenvalues of the closed-loop system are equal to -1.

Solution

$$A - BK = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & \beta_1 & & & \\ & & 0 & 1 & & \\ & & & -k_1\beta_1 & 0 & \\ & & & & 0 & \beta_2 \\ & & & & & -k_2\beta_4 \end{bmatrix}$$

It can clearly be seen that if $k_1 = \frac{1}{\beta_3}$ and $k_2 = \frac{1}{\beta_4}$ the closed-loop system will have two of its diagonal entries equal to -1, implying two of the closed-loop eigenvalues will be -1.

g) Is the closed-loop system found in part(f) stable? Explain your answer. Describe in words how the closed-loop system would respond to the initial condition

$$\delta x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}.$$

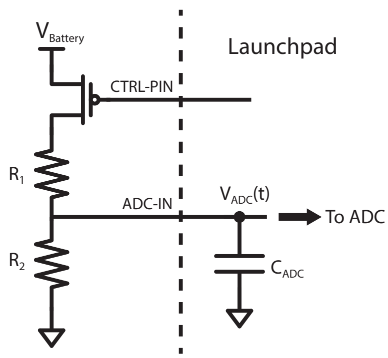
Solution

No, the system is still marginally stable, since 4 of the 6 closed-loop eigenvalues are 0. With a positive perturbation in the angular velocity (4th dimension), the closed-loop linearized system would restore the angular velocity to 0, but not before a positive angle displacement is incurred (3rd dimension). This positive angle displacement would cause the lateral velocity (2nd dimension) of the system to increase at a constant rate, which would in turn cause the lateral position (1st dimension) to increase quadratically with time. Therefore the linearized model of the quadrotor would continually accelerate away from the point $y = 0$.

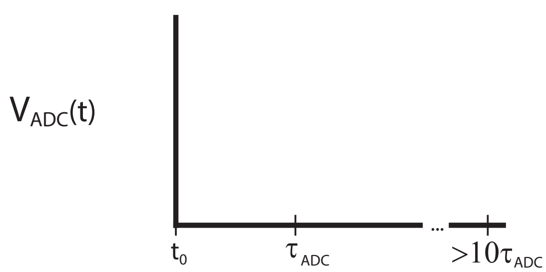
4 Analog to Digital Converter

An ADC (analog to digital converter) is used to convert an analog signal from the real world, to a digital one that a computer can interpret and act on. Let's examine a couple of ADC uses.

a) You'd like to monitor the battery on your 16B car, so that you know when it's low. Your TA gives you the following switched, voltage-divider circuit. C_{ADC} is used by the launchpad's ADC to read and save the input voltage and PMOS threshold voltage $V_{TH} = V_{Battery}/2$.



i) $CTRL - PIN$ is switched to GND at $t_0 = 0$, after a long period of being at $V_{Battery}$. Draw $V_{ADC}(t)$. Indicate values on the y-axis in terms of $V_{Battery}$ for each indicated x-axis value for the circuit's time constant τ_{ADC}.



Solution

Students should draw a decaying exponential rising from 0 to $V_{Battery}\frac{R_2}{R_1+R_2}$. They should mark $0.63V_{Battery}/3$ for $1\tau_{ADC}$ and approximately $0.63V_{Battery}/3$ for $10\tau_{ADC}$.

ii) We would like to estimate the battery voltage as quickly as possible to keep from wasting power. You are given that $\tau_{ADC} = C_{ADC}(R_1||R_2)$. If the ADC samples at $t_{sample} = 1 \times 10^{-3}$ and $V_{ADC}(t_{sample}) = 2.1\,\text{V}$, what is the battery voltage $V_{Battery}$? You may use any of the values $C_{ADC} = 100 \times 10^{-9}\,\text{F}$, $R_1 = 20 \times 10^3\,\Omega$, $R_2 = 10 \times 10^3\,\Omega$, PMOS switch resistance $R_p = 1\,\Omega$, and PMOS gate capacitance $C_p = 1 \times 10^{-12}\,\text{F}$.

Solution

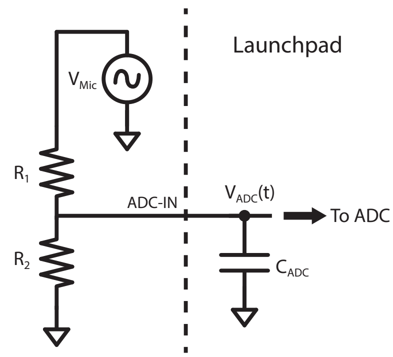
We know that a charging RC circuit can be described as:

$$V_{ADC} = V_{MAX}(1 - e^{-t_{sample}/\tau_{ADC}})$$

Where from given values and voltage divider ratio $V_{MAX} = V_{Battery}\frac{R_2}{R_1+R_2} = \frac{1}{3}$, and we are given $\tau_{ADC} = C_{ADC}(R_1||R_2) = \frac{2}{3} \times 10^{-3}$.

$$\begin{aligned}
V_{ADC} &= \frac{V_{Battery}}{3}(1 - e^{-3/2}) \\
2.1\,\text{V} &= \frac{V_{Battery}}{3}0.7769 \\
V_{Battery} &= \frac{2.1 * 3}{0.7769} \\
V_{Battery} &= 8.1\,\text{V}
\end{aligned}$$

b) You want to try a fancier microphone than the mic-board to record your audio data with. But the new microphone outputs signals from 0 to 9V! Your lab mate suggests you use the circuit you used in part a to monitor the battery circuit.



i) Symbolically find the factored form transfer function $H(j\omega) = \frac{V_{ADC}}{V_{Mic}}$. What is the corner frequency ω_c of this transfer function in terms of symbols R_1, R_2, and C_{ADC}?

Solution

Recognize that the schematic is a voltage divider between impedances R_1 and $Z_{C_{ADC}}||R_2$.

$$Z_{C_{ADC}}||R_2 = \frac{\frac{1}{j\omega C_{ADC}} R_2}{\frac{1}{j\omega C_{ADC}} + R_2}$$

$$\begin{aligned}
H(j\omega) = \frac{V_{out}}{V_{in}} &= \frac{Z_{C_{ADC}}||R_2}{(Z_{C_{ADC}}||R_2) + R_{\text{TA}}} \\
&= \frac{\frac{\frac{1}{j\omega C_{ADC}} R_2}{\frac{1}{j\omega C_{ADC}} + R_2}}{\left(\frac{\frac{1}{j\omega C_{ADC}} R_2}{\frac{1}{j\omega C_{ADC}} + R_2}\right) + R_1} \\
&= \frac{\frac{R_2}{j\omega C_{ADC}}}{\frac{R_1}{j\omega C_{ADC}} + R_1 R_2 + \frac{2}{j\omega C_{\text{TA}}}} \\
&= \frac{R_2}{R_1 + j\omega C_{ADC} R_1 R_2 + R_2} \\
&= \frac{R_2}{(R_1 + R_2) + j\omega C_{ADC} R_1 R_2}
\end{aligned}$$

$$H(j\omega) = \frac{\frac{R_2}{R_1+R_2}}{1 + j\omega \frac{C_{ADC} R_1 R_2}{R_1+R_2}}$$

$$H(j\omega) = \frac{\frac{R_2}{R_1+R_2}}{1 + j\omega C_{ADC}(R_1||R_2)}$$

ii) If the highest frequency a human voice can produce is 200Hz, what frequency must the ADC sample at to accurately record your voice?

Solution

Full points for writing the Nyquist criterion or $f_s > 2 * 200Hz$ specifying any frequency which is strictly greater than.

iii) Your air conditioner is loud and produces an audio tone at $f_{AC} = 500Hz$. You need to attenuate f_{AC} by at least $\frac{1}{10}$ without significantly attenuating your voice. For the values $C_{ADC} = 100 \times 10^{-9}$ F, $R_1 = 20 \times 10^3\ \Omega$, $R_2 = 10 \times 10^3\ \Omega$, show that $H(j\omega)$ from part $b.i.$ is insufficient to meet these requirements.

Solution

The corner frequency is $\omega_c = \frac{1}{C_{ADC}(R_1 || R_2)}$. Plugging in values and converting to Hz gives us $f_c = 240Hz$. The low pass corner is above the signal frequency, so the signal is not significantly attenuated. However, we must attenuate the noise by 10, and the first order low pass does not attenuate signals by 10 until 2400Hz. So a first order filter cannot possibly meet our spec.

iv) Propose a new $H'(j\omega)$ in factored form that would meet the requirements in part b.iii. Specify the frequency of the the poles or zeros you use. You do not need to design a schematic to implement the filter.

Solution

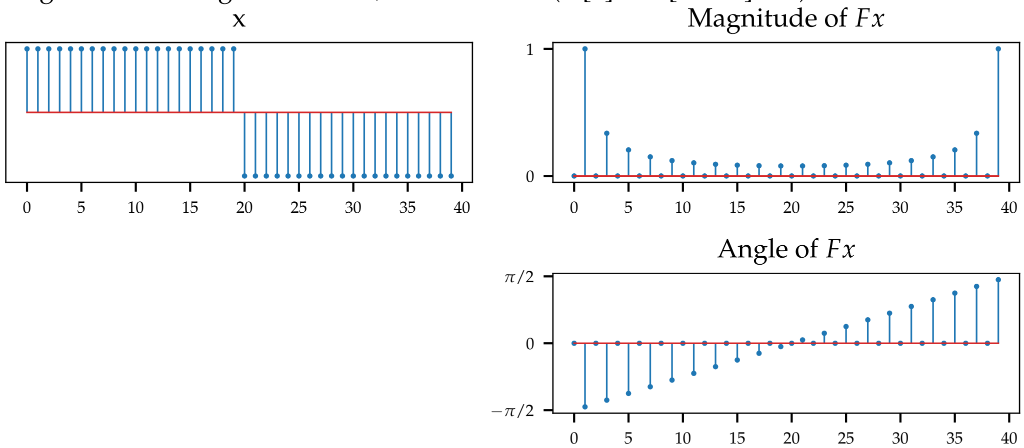
If we take the magnitude of $H(j\omega)$ with any corner frequency $\omega_c = 200 - 260Hz$, we'll need a third or higher order filter.

$$H(j\omega) = \frac{1/3}{(1 + j\frac{\omega}{\omega_c})^3}$$

Estimates using $(\frac{\omega}{\omega_c})^n$ will find they need a 4th order filter which is also fine.

5 Convolution limit

The vector $x \in \mathbb{R}^{40}$ is a sample of a square wave, as shown below. Shown also are the magnitude and angle of $X = Fx$, the DFT of x. ($X[1] = X[N-1] = 1$)



Let x^r denote the rth convolution of x with itself:

$$x^1 = x, \quad x^2 = x * x, \quad \ldots, \quad x^r = x \underbrace{* x * \ldots * x}_{r-1 \text{ times}}.$$

a) Give a formula for $F(x^r)[k]$, the kth DFT coefficient of x^r, in terms of $X[k]$.

Solution

It is $\sqrt{N}^{r-1} X[k]^r$. This follows from the DFT convolution property.

b) Let x^∞ be the normalized limit of x^{4r}:

$$x^\infty = \lim_{r \to \infty} x^{4r}.$$

$X^\infty = F(x^\infty)$ has only two nonzero components, and they are equal. What are they?

Solution

This problem had a bug (my bad). I meant X^∞ to converge to 1 at frequencies ± 1, and 0 elsewhere (communicated quite ineffectively by "normalized limit"). This answer and subsequent progress received full credit.

Actually this limit doesn't exist as the factor of $\sqrt{N}^{r-1}$, which I forgot to account for when designing the problem, increases without bound and dominates X. Any answer that pointed out that the limit doesn't exist effectively received full credit for this part and the next.

c) Argue that x^∞ has the shape of a sinusoid.

Solution

Again, the limit technically doesn't exist (my bad). But the answer hinted at in the problem, which is that up to scaling, $x \to u_1 + u_{39}$, received full credit.