EECS 16B Designing Information Devices and Systems II
Summer 2020 UC Berkeley

SVD Review

1. Pick a Matrix

Given an example of a square matrix that satisfies the following conditions or prove that no such example can exist.

(a) (i) Can be diagonalized and is invertible.
(ii) Cannot be diagonalized but is invertible.
(iii) Can be diagonalized but is non-invertible.
(iv) Cannot be diagonalized and is non-invertible.

(b) (i) Has orthogonal columns and is invertible.
(ii) Has orthogonal columns but is non-invertible.
(iii) Has orthonormal columns and is diagonalizble.

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2. Orthonormal T/F

(a) ___ If U is a matrix with orthonormal columns, then $U^*U = I$.

(b) ___ If U is a matrix with orthornomal columns, then $UU^* = I$.

(c) ___ If U is matrix with orthonormal columns then $\|U\vec{x}\| = \|\vec{x}\|$ for all $\vec{x} \in \mathbb{C}^n$.

(d) ___ A matrix U with orthonormal columns has real eigenvalues.

(e) ___ The singular values of a unitary matrix are all equal to 1.

(f) ___ The eigenvalues of a unitary matrix are all equal to 1.

3. Spectral T/F

(a) ___ The matrix A^*A is Hermitian.

(b) ___ A symmetric matrix can have complex eigenvalues.

(c) ___ The matrix A^*A has positive eigenvalues.

(d) ___ For a Hermitian matrix, the eigenvectors of distinct eigenvalue are orthogonal.

(e) ___ Linearly independent eigenvectors of the same eigenvalue of a Hermitian matrix are orthogonal.

(f) ___ The U and V matrices of the SVD of a Hermitian matrix are identical.

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4. SVD Stuff (X pts)

(a) Compute the SVD of the following matrix. Express your answer in the the form of $\sum_i \sigma_i \vec{u}_i \vec{v}_i^\top$

$$A = \begin{bmatrix} \vec{a} & -\vec{a} \end{bmatrix}$$

Here, $\vec{a}$ is some arbitrary vector in $\mathbb{R}^n$

(b) Compute the compact form SVD of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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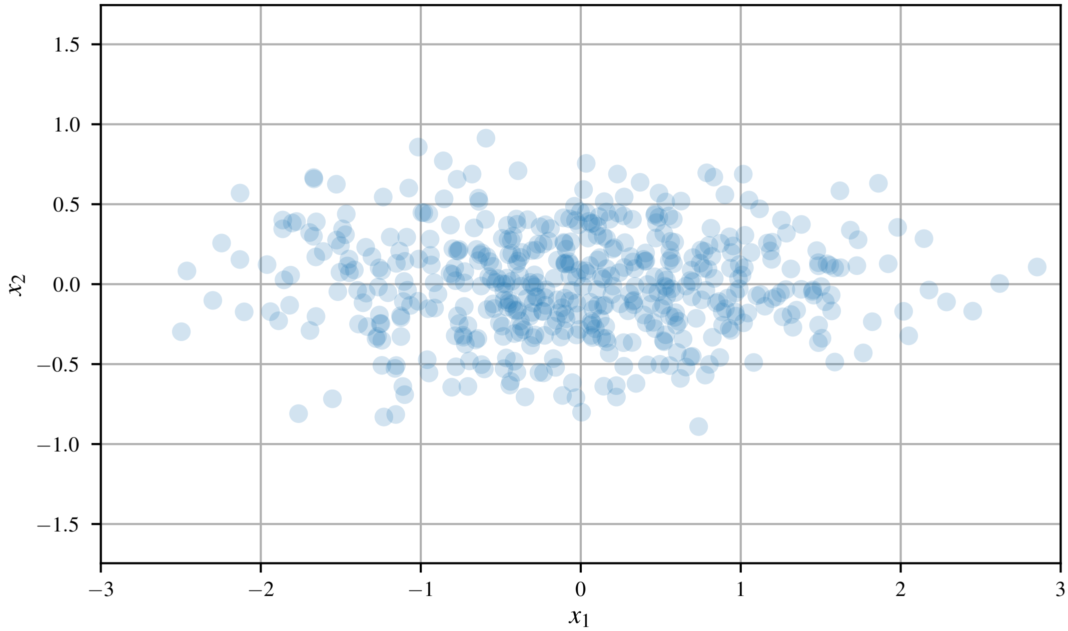
(c) Consider an A matrix where each row is a vector in $\mathbb{R}^2$ that corresponds to one point in the plot below:



On the plot above, draw your best estimate for $\vec{v}_1$ and $\vec{v}_2$.

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5. Spectral Norm Proof

How can we measure the size of a matrix? One way to think about this is to look at the ratio $\frac{||A\vec{x}||}{||\vec{x}||}$ over all vectors $\vec{x}$. In fact, the **Spectral Norm** of a matrix $\|A\|_2$ can be defined as

$$\|A\|_2 = \max_{\vec{x}\neq\vec{0}} \frac{||A\vec{x}||}{||\vec{x}||}$$

In this problem we will try to find what the value of $\|A\|_2$ is and show that it in fact is related to the SVD. Let's start by noting that $A^T A$ is symmetric and has eigenvalue decomposition $V\Lambda V^T$.

(a) **For $\vec{x} \in \mathbb{R}^n$, decompose $\vec{x}$ as a linear combination of the set of orthonormal eigenvectors of $A^T A$.**

(b) **Express $||A\vec{x}||^2$ in terms of $\vec{v}_i$, $\vec{x}$, and σ_i**, for $i \in \{1, 2, ..., n\}$.

(c) **Find a unit vector $\vec{x}$ that maximizes $||A\vec{x}||^2$**.

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(d) **Show that** $\|A\vec{x}\| \leq \sigma_1 \|x\|$ **for any** $x \in \mathbb{R}^n$**. Thus,** $\|A\|_2 = \sigma_1$.

(e) **Show that** $\|A\vec{x}\| \geq \sigma_n \|x\|$ **for any** $x \in \mathbb{R}^n$**. Thus,** $\min \frac{\|A\vec{x}\|}{\|x\|} = \sigma_n$.

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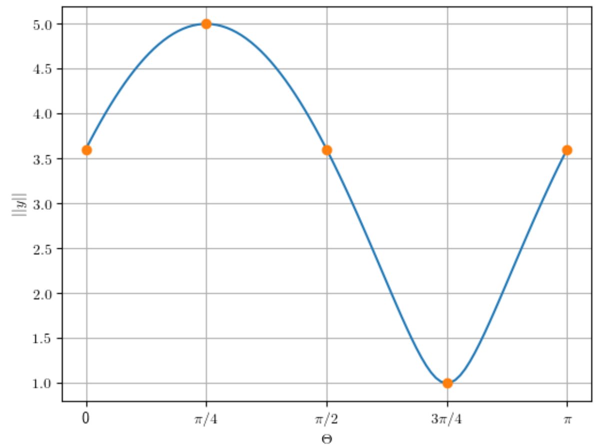
6. SVD (X points)

(a) Let $A \in \mathbb{R}^{2\times 2}$ and $\vec{x} = \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \end{bmatrix}$, $||\vec{x}|| = 1$. Now let $\vec{y} = A\vec{x}$. Below is the plot of $||\vec{y}||$ vs θ.



A has the SVD $U\Sigma V^\top$. Either specify what the matrices U, Σ, and V are; or state they they cannot be determined from the information given.

(b) Let $A \in \mathbb{R}^{N\times N}, B \in \mathbb{R}^{N\times N}$ be full rank matrices and let $\vec{x} \in \mathbb{R}^N$ have $||\vec{x}|| = 1$. Let $\vec{y} = AB\vec{x}$.
Find a upper bound for $||\vec{y}||$ in terms of the singular values of A and B. Explain your answer.

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7. Low Rank Approximation

Given a $m \times n$ matrix A, of high rank, we want to see how we can best approximate this matrix A using a lower rank matrix A_k of rank $k << n$.

To measure this Low-Rank approximation, we will look at the following norm: $\|A - B_k\|_2$ where B_k is a matrix of rank k. In this problem, we will are interested in the following optimization problem

$$\min_{B_k} \|A - B_k\|_2$$
$$\text{subject to } \text{Rank}(B_k) \leq k$$

and show that the optimal B_k is in fact A_k or the rank k SVD approximation of A:

$$A_k = \sum_{i=1}^{k} \sigma_i \vec{u}_i \vec{v}_i^T \tag{1}$$

(a) **What is the spectral norm of** $\|A - A_k\|_2$?

To show A_k is optimal, we must show that $\|A - B_k\|_2 \geq \|A - A_k\|_2 = \sigma_{k+1}$ for any matrix B_k of rank k.

To do this, we will first consider a vector $\vec{y} = \alpha_1 \vec{v}_1 + \cdots + \alpha_{k+1} \vec{v}_{k+1}$ that is a linear combination of the first $k+1$ vectors of the V matrix of the SVD of A. We will also define a subspace $S = \text{span}\{\vec{v}_1, \ldots, \vec{v}_{k+1}\}$ and show that there must exist a vector $\vec{y}$ in both S and $\text{Nul}(B_k)$.

(b) **What are the dimensions of the Nul**(B_k) **and** S?

(c) **Show that there must exist a** $\vec{y} = \alpha_1 \vec{v}_1 + \ldots + \alpha_{k+1} \vec{v}_{k+1}$ **that is in both subspaces Nul**(B_k) **and** S. *Hint: Let R be a basis for S. Then create a basis B for the Nul(B_k) and look at the union of the two bases.*

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(d) Let $\|\vec{y}\| = 1$ and **show that** $\|A - B_k\|_2 \geq \left\|(A - B_k)\vec{y}\right\|$.

(e) **Express** $\|\vec{y}\|^2$ **and** $\|A\vec{v}_i\|^2$ **in terms of** $\alpha_1, \ldots, \alpha_{k+1}$.

(f) **Simplify** $\left\|(A - B_k)\vec{y}\right\|$ **and conclude that** $\|A - B_k\|_2 \geq \sigma_{k+1}$.

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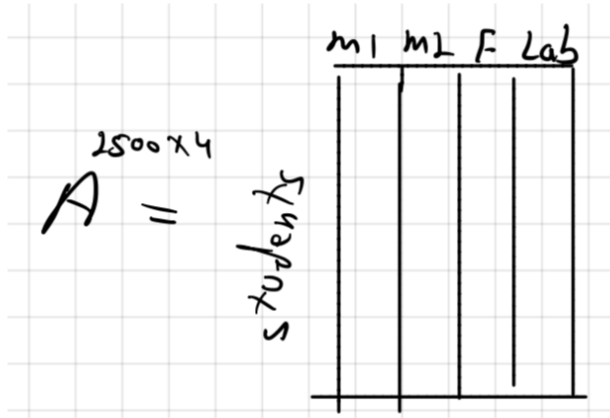
8. PCA Midterm question

This question comes from fa17 midterm 2.

Consider a matrix $A \in \mathbb{R}^{2500\times4}$ which represents the EE16B Sp'2020 midterm 1, midterm 2, final and lab grades for all 2500 students taking the class.

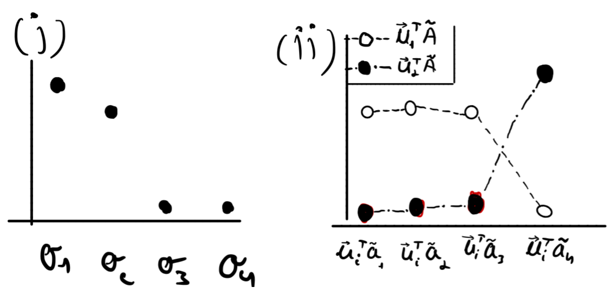


To perform PCA, you subtract the mean of each column and store the results in $\tilde{A}$. Your analysis includes:

(a) Computing the SVD: $\tilde{A} = \sigma_1\vec{u}_1\vec{v}_1^T + \sigma_2\vec{u}_2\vec{v}_2^T + \sigma_3\vec{u}_3\vec{v}_3^T + \sigma_4\vec{u}_4\vec{v}_4^T$ and plot the singular values.

(b) Computing the graph $\vec{u}_1^T\tilde{A}$ and $\vec{u}_2^T\tilde{A}$

The analysis data are plotted below:



Based on the analysis, answer the following <u>true or false</u> questions. Briefly explain your answer.

(a) ___ The data can be approximated well by two principle components.

(b) ___ The students' exam scores have significant correlation between the exams.

(c) ___ The middle plot (ii) shows that students who did well on the exam did not do well in the labs and vice versa.

(d) ___ One of the principle components attributes is solely associated with lab scores and not with exam scores.

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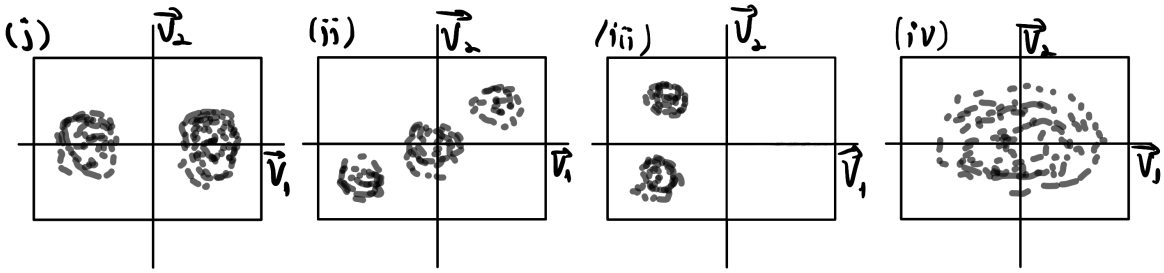
(e) Circle all the scatter plots that could describe the data projected on the largest two principle components.



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