# An Analytical Approach to Broadband Matching

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*Abstract*— **The purpose of this project is to study different broadband matching techniques for microwave amplifiers. Two different methodologies for broadband amplifier design are developed and employed. The first method involves the design of a 6 $dB$/octave roll-up filter to compensate for the 6 $dB$/octave roll-off inherent to the active device. The second method seeks a closed form solution to the gain-matching problem with the use of second order matching networks. For each technique a two-octave gain-compensated amplifier is designed.**

## I. Introduction

THE ideal microwave amplifier would have constant gain and good input matching over the desired frequency bandwidth. Conjugate matching will give maximum gain at the center of the band but results in unpredictable bandwidth and ripple. Another problem is the 6 $dB$/octave roll-off of $|S_{21}|$. Some of the common approaches to this problem are listed below[1] ; in each case an improvement in bandwidth is achieved only at the expense of gain, complexity or other factors.

### A. Commonly used broadband techniques

- Compensated matching networks: Input and output networks can be designed to compensate for the gain roll-off in $|S_{21}|$, but generally at the expense of the input and output match.
- Resistive matching networks: Good input and output matching can be obtained by using resitive matching networks, with a corresponding loss in gain and increase in noise figure.
- Negative feedback: Negative feedback can be used to flatten the gain response of the transistor, improve the input and output match, and improve the stability of the device. Amplifier bandwidths in excess of a decade are possible with this method, at the expense of gain and noise figure.
- Balanced amplifiers: Two amplifer having 90° couples at their input and output can provide good matching over an octave bandwidth, or more. The gain is equal to that of a single amplifier, however, and the design requires two transistors and twice the DC power.
- Distributed amplifiers: Several transistors are cascaded together along a transmission line, giving good gain, and noise figure over a wide bandwidth. The circuit is large, and does not give as much gain as a cascade amplifier with the same number of stages.

### B. This project

The purpose of this project is to study different methodologies to design matching networks that compensate for the 6 $dB$/octave roll-off of $|S_{21}|$. One method consists of network synthesis of a sloped bandpass filter, i.e. a filter with a 6 $dB$/octave roll-up over the passband. The second method is the use of two element matching networks.

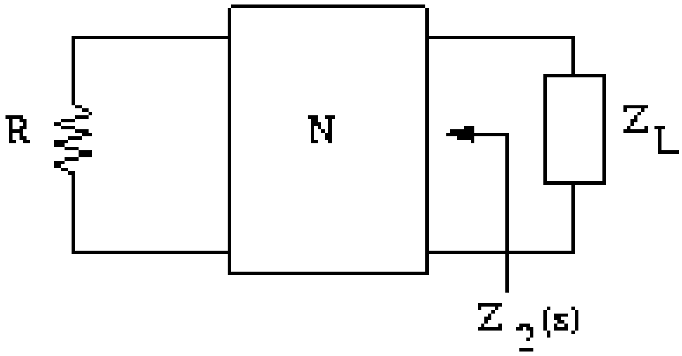



Fig. 1. Broadband matching problem.

## II. Network synthesis of sloped bandpass filters

This theory[2] assumes the transistor to be unilateral, so that the input and output networks can be designed seperately. In order to determine whether a transistor can be considered unilateral from the standpoint of transducer power gain, the unilateral figure of merit $u$ can be used :

$$u = \frac{|S_{11}|\,|S_{22}|\,|S_{12}|\,|S_{21}|}{\left(1-|S_{11}|^2\right)\left(1-|S_{22}|^2\right)} \tag{1}$$

If the transistor can be considered as unilateral, the input can be approximated by a series $RC$-network and the output by a parallel $RC$-network. In the case of discrete transistor, inductors representing leads have to be added.

### A. General lumped broadband matching theory

Consider the broadband matching problem in figure 1, where $N$ denotes a passive lossless matching network that has to match $Z_L$ to $R$. Then

$$G(\omega^2) = |S_{21}(j\omega)|^2 = 1 - |S_{11}(j\omega)|^2 = 1 - |S_{22}(j\omega)|^2 \tag{2}$$

or (using analytic continuation)

$$G(-s^2) = 1 - |S_{22}(s)|^2 \tag{3}$$

One can prove[2] that

$$S_{22}(s) = \frac{Z_2(s) - Z_L(-s)}{Z_2(s) + Z_L(s)} \tag{4}$$

The poles of $S_{22}(s)$ in the right half plane are the poles of $Z_L(-s)$, since the other terms in equation (4) are positive-real and cannot have poles in the right half-plane. Let $s_i \quad (i = 1, 2, \ldots, m)$ denote the the poles of $Z_L(-s)$ (real or complex conjugate pairs). Define $B(s)$ as

$$B(s) = \prod_{i=1}^{m} \frac{s - s_i}{s + s_i} \tag{5}$$

Then $B(s)$ has only poles in the left half-plane, is real for real $s$ and $B(s)B(-s) = 1$ ($B(s)$ is an all-pass function). Define then $\rho(s)$ as

$$\rho(s) = B(s)S_{22}(s) \tag{6}$$

This reflection coefficient $\rho(s)$ has only poles in the left half-plane. Substituting equation (6) in equations (3) and (4) yields

$$Z_2(s) = \frac{2r_L(s)B(s)}{B(s) - \rho(s)} - Z_L(s) \tag{7}$$

$$r_L(s) = \frac{Z_L(s) + Z_L(-s)}{2} \tag{8}$$

$$|\rho(j\omega)|^2 = 1 - G(\omega^2) \tag{9}$$

For $\rho(s)$ to be real, it is necessary that $0 \leq G(\omega^2) \leq 1$. Once $G(\omega^2)$ is given, $\rho(s)$ can be determined by spectral factorisation (and using analytic continuation) :

$$\rho(s)\rho(-s) = 1 - G(-s^2) = \frac{N(s^2)}{M(s^2)} \tag{10}$$

$N(s^2)$ and $M(s^2)$ are respectively the numerator and the denominator of $G(s^2)$, and they both can be decomposed as :

$$N(s^2) = n(s)n(-s) \tag{11}$$

$$M(s^2) = m(s)m(-s) \tag{12}$$

These $n(s)$ and $m(s)$ are formed by the left half-plane zeros of respectively $N(s)$ and $M(s)$. Therefore :

$$\rho(s) = \pm\frac{n(s)}{m(s)} \tag{13}$$

So, given $G(\omega^2)$, $\rho(s)$ can be determined by spectral factorisation. If also $Z_L(s)$ is given, then $Z_2(s)$ can be determined by using equation (8). Under certain conditions[2] for $\rho(s)$, a passive network can be derived that implements $Z_2(s)$.

### B. *Input lossless matching*

Examining the unilateral figure of merit $u$ for the EE217 transistor (figure 6), this transistor can be considered highly non-unilateral (or bilateral). This means first of all that there is a risk of making the transistor unstable. Furthermore the input and ouput matching networks cannot be designed independently.

The reason that $u$ is so high for this transistor, is that $S_{11}$ is so close to 1. This also means that we can get a considerable gain from the input network. Taking all this into account, we decided to add only an input match, that would perform matching and 6 $dB$/octave roll-up at the same time. $G(\omega^2)$ can be specified as :

| | |
|---|---|
| low frequency | 2 $GHz$ |
| high frequency | 8 $GHz$ |
| ripple | 1 $dB$ |
| filter order | 2 |

TABLE I
Broadband amplifier specifications.

$$G((2\pi f)^2) = \frac{K_n \cdot (f/f_c)^2}{1 + \varepsilon^2 C_n\left(\frac{f}{B}\left(\frac{f}{f_0} - \frac{f_0}{f}\right)\right)^2} \tag{14}$$

$f_0$ is the center frequency, $B$ is the bandwidth. The maximum ripple is given by $1/(1+\varepsilon^2)$. $K_n$ and $f_c$ are proportionality constants so that $0 \leq G(\omega^2) \leq 1$ for all frequencies. $C_n(x)$ are the Chebychev polynomials of the first kind :

$$T_0(x) = 1 \tag{15}$$

$$T_1(x) = x \tag{16}$$

$$T_2(x) = 2x^2 - 1 \tag{17}$$

The equivalent input network of the transistor consists of a series $RC$-network. This means that, using the terminology of section II-A,

$$Z_L = R_i + \frac{1}{sC_i} = \frac{1 + sRC_i}{C_i} \tag{18}$$

For the EE217 transistor at high frequencies :

$$R_i = RI + RS = 11.25\ \Omega \tag{19}$$

$$C_i = CGS \quad = 0.31\ pF \tag{20}$$

Using the specifications in table I and going through the steps described in section II-A, $Z_2(s)$ turns out to be :

$$Z_2(s) = \frac{2s^4 + 2.71s^3 + 8.75s^2 + 3.16s + 2}{0.630s^3 + 0.283s^2 + 0.179s} \tag{21}$$

$$= 3.17s + \frac{11.2}{s} + \cfrac{1}{\frac{0.156}{s} + \cfrac{1}{\frac{1.82}{s} + 2.87}} \tag{22}$$

This is a second order chebychev bandpass filter with a 6 $dB$/octave roll-up over the passband. All the numbers are normalized to a center pulsation $\omega_0$ of 1 $rad/s$. The corresponding ciruit is given in figure 2 and the element values are given in table II. Note that $CGS$ fullfills the function of $C_1$, $R_1$ is the input resistance $R_i$ of the transistor. $R_2$ is the source resistance.

The input matching network expects a source resistance $R_2$ of 32.3 $\Omega$. This can be converted to 50 $\Omega$ by an ideal transformer. The ideal transformer can be removed by introducing an extra element ($L_2$) and changing the other element values. Table III gives the final input matching

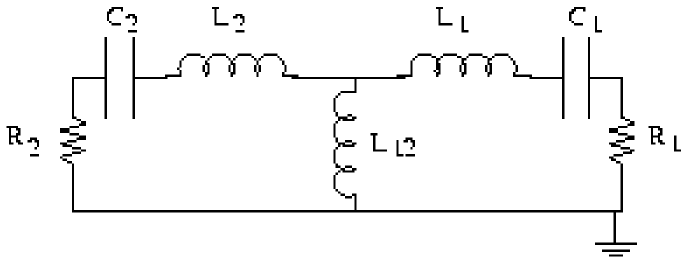



Fig. 2. Second order chebychev bandpass filter with a 6 $dB$/octave roll-up over the passband.

| | |
|---|---|
| $C_1$ | 310 $fF$ |
| $L_1$ | 1.42 $nH$ |
| $L_{12}$ | 2.86 $nH$ |
| $C_2$ | 1.95 $pF$ |
| $L_2$ | 0 $nH$ |
| $R_2$ | 32.3 $\Omega$ |

TABLE II
Broadband matching network.

network. This network can be obtained by equating the $Z$-matrix of the network in table II with an ideal transformer and the $Z$-matrix of a similar network without an ideal transformer but with an additional inductor $L_2$.

The final circuit, including the transistor, is shows in figure 7. Figure 8 shows the simulated results this circuit and confirms the specifications given in table I.

### C. Conclusion

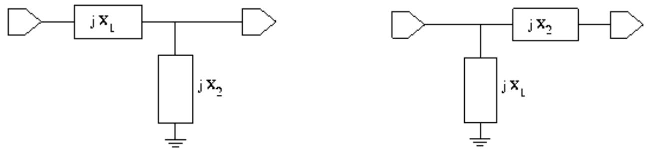
Network synthesis of sloped bandpass filters is an interesting technique to compensate for the 6 $dB$/octave roll-off of the transistor. The major drawback of this method is that it relies on network synthesis for the filter design, which has to be done by hand and which limits its practical implementation for a bandpass chebychev filter to a second order filter (second order equivalent low pass).

## III. Two-Element Matching Networks

In this section, we attempt to derive a closed-form analytical method for determining component values for a 2-element broadband matching network. The matching is performed in order to compensate for gain roll-off in the active device and to present a flattened gain response over a wider frequency range. The matching network is designed

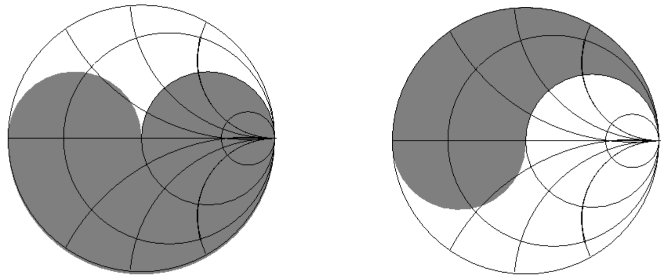
| | |
|---|---|
| $C_1$ | 310 $fF$ |
| $L_1$ | 0.721 $nH$ |
| $L_{12}$ | 3.563 $nH$ |
| $C_2$ | 1.256 $pF$ |
| $L_2$ | 0.871 $nH$ |
| $R_2$ | 50 $\Omega$ |

TABLE III
Transformed broadband matching network.




(a) (b)

Fig. 3. General form of a 2-element $L$-match (a) and $\pi$-match (b) networks.



(a) (b)

Fig. 4. Acheivable impedances of two inductor L-match (a) and $CL$ $\pi$-match (b). The shaded regions correspond to unattainable impedances.

to provide exact gain compensation at the band edges; a byproduct of this process is a flattened gain response over the bandwidth of interest.

### A. Topology Selection

The circuit topology used in the mathematical derivation below is that of a lossless two-element matching network (see figure 3). There are a total of 8 possible two-element matching networks: the $L$-match and the $\pi$-match, each with four possible combinations of $L$ and $C$. The analytical process is quite similar for the $L$-match and the $\pi$-match, so only the $L$-match is shown here.

It should be noted that the choice of matching network is quite important in determining the quality and type of match achieved and should be based on the S-parameters of the active device. For example, the range of achievable impedances of two different matching networks are shown in figure 4. The matching topology should be selected such the required gain circles lie within the achievable impedance region of the network.

### B. Component Selection

Once an appropriate topology is chosen for the input and output networks, the analytical model can be derived. As an example, a two inductor L-match of figure 3(a) is presented. It is assumed that the required gain circles at the frequency band edges have already been calculated and are given as input parameters to this analysis[1]. Each gain

[1] If not, generating the gain circles is a straightforward process as described in [1].

circle can be represented by its complex center position, $C_r + jC_i$, and its radius, $R$. Therefore, we can introduce the unknown parameter $\theta$, and express the locus of the gain circle as:

$$\Gamma_c = C_r + jC_i + R \cdot e^{j\theta} \tag{23}$$

The locus of the gain circle is converted normalized impedance, $Z_c$, by the standard transformation.

$$Z_c = \frac{1+\Gamma_c}{1-\Gamma_c} \tag{24}$$

$Z_c$ can then be separated into its resistive and reactive components, $R_c$ and $X_c$, respectively:

$$R_c = \frac{1 - C_r^2 - C_i^2 - R^2 - 2R(C_r \cos\theta + C_i \sin\theta)}{\left|1 - C_r - jC_i - Re^{j\theta}\right|^2} \tag{25}$$

$$X_c = \frac{2(C_r + R\sin\theta)}{\left|1 - C_r - jC_i - Re^{j\theta}\right|^2} \tag{26}$$

The input impedance of the loaded matching network, $Z_{in}$, can also be calculated and separated into its resistive and reactive components, $R_{in}$ and $X_{in}$. For the case of the two inductor L-match :

$$R_{in} = \frac{4\pi^2 f^2 L_2^2}{1 + 4\pi^2 f^2 L_2^2} \tag{27}$$

$$X_{in} = 2\pi f L_1 + \frac{2\pi f L_2}{1 + 4\pi^2 f^2 L_2^2} \tag{28}$$

In order to achieve perfect gain compensation at the band edges, one must set $R_c = R_{in}$ and $X_c = X_{in}$ at both band edges, resulting in four equations dependent on $L_1$, $L_2$, $\theta_1$ and $\theta_2$. In order to reduce the complexity of these four equations, several algebraic steps are taken. First, the numerators and denominators of the equations are cross multiplied, and expressions of the form $|a + jb|^2$ are replaced by $a^2 + b^2$. Then, several trigonometric identities are employed in order to remove the variables $\theta_1$ and $\theta_2$ from the equations. These four equations reduce to two equations of the form:

$$a_1 + a_2 L_1 + a_3 L_2 + a_4 L_1^2 + a_5 L_2^2 + a_6 L_1 L_2 + \cdots \\ a_7 L_1 L_2^2 + a_8 L_1^2 L_2^2 = 0 \tag{29}$$

where $a_1$ to $a_8$ are functions of frequency and the gain circle parameters mentioned above.

Therefore, solutions for $L_1$ and $L_2$ can be found from these 2 equations. Unfortunately, such a form cannot be solved in analytic form. Using numerical methods, however, acceptable values of $L_1$ and $L_2$ can be found on a case-by-case basis. This process can be repeated for both the input and output matching networks to achieve the desired gain matching at the band edges, and hence a flattened gain response over the frequency band.

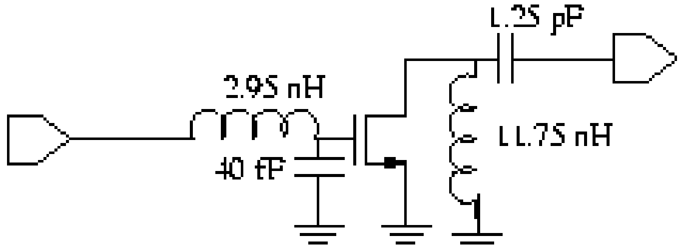



Fig. 5. Final topology of broadband amplifier.

### C. Practical Example

The above approach was used to design a broadband amplifier with a nominal gain of 14 $dB$ and a passband ripple of 1.8 $dB$ over a bandwidth of 1.2 - 5.4 $GHz$. The transistor used was a unilateral version of the EE217 FET, as it was determined that the bilateral transistor was not sufficiently stable for this compensation technique.

The final topology is shown in figure 5, and the gain response is shown in figure 9. The sharp roll-off at low frequencies is due to a second-order high-pass pole at 1.31 $GHz$ in the output matching network.

### D. Extensions and Limitations

The method outlined above can be used to generate two-element gain matching networks for use in broadband amplifier design. Also with certain extensions, this approach is general enough to be applied to other broadband problems which require "design circles", such as stability circle or noise figure circle design. For instance, this approach can be directly applied to noise figure circles instead of gain circles. The designer can choose a desired noise figure at each of the band edges and use the procedure above to determine an acceptable input matching network. Then the designer can check the mid-band region to make sure the noise figure remains within bounds and iterate as necessary.

A major limitation of this model is its restriction to two-element matching networks. As a result, the achievable impedances of the matching network are severely limited. Also, the impedance of a two-element match moves clockwise on the Smith chart as frequency increases, whereas the locus of the gain circles tend to move counterclockwise with increasing frequency. Hence, a two-element match is not well-suited to optimum gain matching over a broad bandwidth.

## References

[1] David M. Pozar, *Microwave Engineering*, John Wiley & Sons, second edition, 1998.
[2] Tri T. Ha, *Solid-State Microwave Amplifier Design*, John Wiley & Sons, 1981.