EE221A Section 2

September 4, 2020

Based on Lectures 2 and 3

Spaces, functions, vector spaces, linear independence, linear maps, dual and range spaces

1 Functions

Definition 1. \( f : X \to Y \) is said to be a function if, for all \( x \), \( f \) assigns a unique \( y = f(x) \) in \( Y \).

Definition 2. \( f : X \to Y \) is said to be

- injective if
  \[ \forall x_1, x_2 \in X, \quad x_1 \neq x_2 \implies f(x_1) \neq f(x_2). \]
- surjective if
  \[ \forall y \in Y, \exists x \in X \text{ such that } y = f(x). \]
- bijective if it is both injective and surjective.

Problems 1 (Maps and functions):

(a) Can you find a function, which is injective but not surjective?
(b) Can you find a function, which is surjective but not injective?
(c) Can you find a function, which is bijective?
(d) Can you find a function, which is neither injective nor surjective?

group work: write on (front)

2 Vector Spaces

Definition 3. \( V \) is a vector space over a field \( F \) if there exist two binary operations: vector addition \((\cdot + V \times V \to V)\) and scalar multiplication \((\cdot : F \times V \to V)\) such that

- (Addition) \( \forall v, w, x \in V \)
  \[ v + (w + x) = (v + w) + x \]
  \[ v + w = w + v \]
  \[ \exists 0 \in V \text{ such that } v + 0 = v \]
- (Scalar multiplication) \( \forall \alpha, \beta \in F, \forall v \in V \)
  \[ \alpha(v + w) = \alpha v + \alpha w \]
  \[ (\alpha + \beta)v = \alpha v + \beta v \]
  \[ \alpha(\beta v) = (\alpha \beta) v \]
  \[ 1 v = v \]

where \( 0 \) is the additive identity of \( V \) and \( 1 \) is the multiplicative identity of \( F \).

Problems 2 (Minimal definitions for vector spaces):

(a) What is a vector space? Give an example of a vector space.

(b) What is a function space? Give an example of a function space.

(c) What is a linear map? Give an example of a linear map.

(d) What is a basis? Give an example of a basis.

Problems 3 (Function spaces):

(a) Show that \( \mathbb{R} \) is a vector space under the usual addition and scalar multiplication.

(b) Let \( V \) be a vector space. Show that \( \mathbb{R} \times V \) is a vector space under the operations defined as:

\[ (a, v) + (b, w) = (a + b, v + w) \]
\[ c(a, v) = (ca, cv) \]

Problems 4 (Conjugate of fields): Prove that \( (\mathbb{C}, \mathbb{R}) \) is not a vector space.

Problems 5 (Linear independence):

(a) Show that \( \{1, x, x^2\} \) is a linearly independent set.

(b) Show that \( \{1, x, x^2, x^3\} \) is a linearly dependent set.

(c) Show that \( \{1, x, x^2, x^3\} \) is a linearly independent set.

Problems 6 (Linear transformations):

(a) Find the matrix representation of the linear transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( T(x, y) = (2x - y, x + 2y) \).

(b) Find the kernel and the image of the linear transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( T(x, y) = (x - y, 2x + y) \).

(c) Find the matrix representation of the linear transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( T(x, y) = (x + y, x - y) \).

(d) Find the matrix representation of the linear transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( T(x, y) = (x - y, x + y) \).

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3 Subspaces

Definition 3. If \( A \) is a subspace of a vector space \( V \) over a field \( F \), then:
1. \( a, b \in A \Rightarrow a + b \in A \)
2. \( a \in A \Rightarrow ka \in A \)
3. \( \emptyset \neq A \subset V \)

Problem 5 (The zero vector). Can you find a subspace of \( V \) that does not have the zero vector of \( V \)?

Problem 6 (Complement of a subspace). Let \( W_1 \) and \( W_2 \) be subspaces of a vector space \( V \). Prove that:
\[ W_1 \cap W_2 \subsetneq V \]

4 Linear Independence

Let \( W \subset V \) be a vector space.

Definition 6. A set of vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) in \( V \) is said to be linearly independent if:
\[ c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n = \mathbf{0} \Rightarrow c_1 = c_2 = \cdots = c_n = 0 \]

Definition 7. A set of vectors \( B = \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \} \) is said to be a basis for \( V \) if:
1. \( B \) is linearly independent.
2. \( \text{span}(B) = \{ \mathbf{v} \in V : \mathbf{v} = \sum c_i \mathbf{v}_i, \text{ for some } c_i \in F \} \)

Problem 7 (Linear Independence). Let \( V = \mathbb{R}^2 \) be a vector space over \( \mathbb{R} \). Consider the matrix:
\[ A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

Are the columns of \( A \) linearly independent? Write a basis for \( V \). Is it unique? Can you find a basis \( B \) with \( \operatorname{rank}(A) = 0 \)?

Problem 8 (Null Space). Given a linear map \( L : U \to V \), we call \( \ker(L) = \{ u \in U : L(u) = \mathbf{0} \} \) the null space of \( L \).

Definition 8. A linear map \( L : U \to V \) is called injective (one-to-one) if \( \ker(L) = \{ \mathbf{0} \} \).

The null space of a linear map \( A \) is:
- the set of vectors mapped to zero by \( y = Ax \)
- the set of vectors orthogonal to all rows of \( A \)
- the range of \( x \) where \( y = Ax \)
- the row space of \( A \)
- the columns \( x \) where \( y = Ax \)
- the set of all solutions to \( y = Ax \)

For a linear map \( A \), a linear map \( A \) is called injective (one-to-one) if \( A \) has a left inverse.

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