EE221A Section 6

October 2, 2020
Based on Lectures 6 and 7

Topics: E and U Theorem of Diff Eqs, BG Lemma, Dynamical Systems, Intro to LTI systems

1 Fundamental Theorem

Theorem 1 (Fundamental Theorem of Differential Equations). Consider the following ordinary differential equation (ODE):

\[
\begin{align*}
\dot{x} &= f(x, t), \\
x(t_0) &= x_0,
\end{align*}
\]

with the vector field \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \). If \( f \) is

- piecewise continuous in \( t \)
- Lipschitz continuous in \( x \),

then the ODE admits a unique solution, which is differentiable almost everywhere except at points where \( f \) is discontinuous with respect to \( t \).

\[
\exists! x(t) \quad \Rightarrow \quad \exists! \Phi(t, t_0) \quad \forall t \geq t_0
\]

2 Bellman-Gronwall lemma

Theorem 2 (Bellman-Gronwall Inequality). Let \( u(\cdot) \) be a nonnegative, piecewise continuous function on \([0, T]\). If

\[
u(t) \leq c_1 + \int_{t_0}^{t} k(\tau)u(\tau)d\tau
\]

for some constant \( c_1 \geq 0 \) and a nonnegative integrable function \( k \), then

\[
u(t) \leq c_1 \exp\left(\int_{t_0}^{t} k(\tau)d\tau\right)
\]

for \( 0 \leq t_0 < t \leq T \).

Sign of \( B-G \) lemma? we get a bound on \( u \) indep of \( u \)!
Problem 1. (Variation on linear systems) Consider the following linear system:
\[
\begin{align*}
x(t) &= Ax(t), \quad t \in (0, T] \\
x(0) &= x_0,
\end{align*}
\]
where the matrix \( A \) is in \( \mathbb{R}^{n \times n} \). Now we consider the variation \( x_0 + \delta x_0 \) of the initial value, and the corresponding linear system:
\[
\begin{align*}
\dot{x} &= A\delta x(t), \quad t \in (0, T] \\
\dot{x}(0) &= \delta x_0.
\end{align*}
\]

Let \( \delta x := \delta x - x \) be the variation on the state. Then \( \delta x \) solves the following linear system:
\[
\begin{align*}
\dot{\delta x}(t) &= A\delta x(t), \quad t \in (0, T] \\
\delta x(0) &= \delta x_0.
\end{align*}
\]

Show that \( \|\delta x(t)\| \to 0 \) as \( \|\delta x_0\| \to 0 \) for any \( t \in [0, T] \) using Bellman-Gronwall lemma.

Significance? Small perturb of IC causes small perturb of trajectory.

\[ \text{Exercise.} \quad \text{Derive the more general Bellman-Gronwall lemma, i.e. } C_1 \text{ is not a constant but rather a set piecewise function in } t, C_1(t). \]

Problem 2. (Differential version of B.G. lemma) Let \( x(\cdot) \) be a nonnegative, continuously differentiable function on \([0, T]\), which satisfies
\[
x(t) \leq a(t)x(t) + b(t)u(t)
\]
for all \( t \in [0, T] \), where \( a, b, \) and \( u \) are nonnegative integrable functions on \([0, T]\). Show that the solution \( x(t) \) is upper bounded by a growing exponential.

\[
\begin{align*}
x(t) &\leq x(0) e^{\int_{0}^{t} a(s)x(s)\,ds} + \int_{0}^{t} b(s)u(s)\,ds \\
&\leq c(0)e^{\int_{0}^{t} x(s)\,ds} \quad \text{where } k(s) \text{ many integrable} \\
&\leq c(t) + \int_{0}^{t} k(s)x(s)\,ds
\end{align*}
\]

October 2, 2020
3 Dynamical Systems

(\(U, Y, \Sigma, \sigma, \tau\)): (input, state, output, state transition function, output read-out map).

- **Input**: \(U \subseteq \{ x : \mathbb{R} \rightarrow \mathbb{R} \} \); \(U\) vector space (typically \(\mathbb{R}\)). (Note that \(U\) is a function space.)

- **Output**: \(Y \subseteq \{ y : [t, \infty) \rightarrow \mathbb{R} \} \); \(Y\) vector space (typically \(\mathbb{R}\)). (Note that \(Y\) is a function space.)

- **State Space**: \(\Sigma\), a vector space (typically \(\mathbb{R}\)).

- **State transition function**: \(\sigma : \mathbb{R} \times \mathbb{R} \times \Sigma \times U \rightarrow \Sigma\) with \(\sigma(t, \phi, \xi, w) = \xi(t)\).

- **Output read-out map**: \(\tau : \mathbb{R} \times \Sigma \times U \rightarrow Y\) with \(\tau(t, \xi, w) = y(t)\).

- **Response function**: composition of \(\sigma\) and \(\tau\); \(\sigma : \mathbb{R} \times \mathbb{R} \times \Sigma \times U \rightarrow \mathbb{R}\) with \(\rho(t, \phi, \xi, w) = y(t)\).

\[
\rho(t) = r(s(t, \phi))
\]

Consider. Why are we defining these vector spaces?

**Problem 3.** Suppose that the dynamical system

\[
\begin{align*}
x(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) + D(t)u(t)
\end{align*}
\]

admits the unique solution

\[
x(t) = \Phi(t, \tau)x_0 + \int_{\tau}^{t} \Phi(t, \zeta)B(\zeta)u(\zeta) \, d\zeta
\]

for \(t \in [t_0, \infty)\). Identify the state transition function, the output read-out map, the response function, zero input-response (natural), and zero-state response (forced).

**Two axioms**

1. **State transition axiom**: given \(u_1, u_2 \in U\) with \(u(t) = u_1(t)\) for \(t \in [t_1, t_2]\), we have

\[
s(t_2, t_1, x_0, u(t_1, t_2)) = s(t_2, t_1, x_0, u(t_1, t_2))
\]

2. **Semi-group axiom**: \(t_0 \leq t_1 \leq t_2, \forall x_0 \in \Sigma, \forall u \in U,

\[
s(t_2, t_0, x_0, u(t_0, t_2)) = s(t_2, t_1, s(t_1, t_0, x_0, u(t_0, t_1))u(t_0, t_1))
\]
4 Linear and Time-Invariant Dynamical Systems

Definition 3. $(U, \Sigma, \Lambda, \chi, \tau)$ is said to be a linear dynamical system if
- $U, \Sigma, \Lambda$ are vector spaces over the same field;
- the response map $\rho$ is linear in both $\alpha_0$ and $\alpha_0$, i.e.,
  \[ \rho(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = \alpha_1 \rho(\alpha_0, \alpha_1, \alpha_2, \alpha_3) + \alpha_2 \rho(\alpha_0, \alpha_1, \alpha_2, \alpha_3). \]

Definition 4. $(U, \Sigma, \Lambda, \chi, \tau)$ is said to be a time-invariant dynamical system if
- We define a shift operator $T_\tau : \mathcal{F} \to \mathcal{F}$ for $\mathcal{F} = U$ or $\Sigma$ such that
  \[ (T_\tau f)(t) = f(t - \tau) \]

- $T_\tau$ and $\Sigma$ are closed under $T_\tau$ for all $\tau$.
- For all $t_0, t_1 \geq t_0, t_1 \in \mathbb{R}$, for all $x_0 \in \Sigma$, for all $u \in U$
  \[ \rho(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = \rho(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)(T_\tau) \]

Consider: (Roller coaster) Consider a system that captures the motion of an empty roller coaster on an a track at an amusement park. Let the dynamical system tracks be the coaster's position and velocity, and let $\tau_0$ be the time that the roller coaster is launched. Is this system time-invariant?

Continuous time-varying linear system

\[ \dot{x}(t) = A(t) x + B(t) u \]
\[ y(t) = C(t) x + D(t) u \]
\[ x(t_0) = x_0, \]

where $x(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, and $A, B, C, D$ are matrices whose elements are functions of $t$.

Problem 4. Is the following system linear? $x(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}$, $x(t) = a(t)$ and $x_0, x_1, t \in \mathbb{R}$

\[ x(t) = x(t) + e^{A(t)} x_0 \]
\[ x_1 = x(t_0) + e^{A(t_0)} x_0 \]
\[ x_2 = e^{A(t_1)} x_0 \]
\[ x(t) = x(t_0) e^{A(t_0)} \]

SYS is NL transfer

If had $a(t)$, instead of $a(t) x_0$, linear $\tau$-can fit to LTI form

\[ x(t) = Ax(t_0) + \int_{t_0}^{t} B(t) u(t) \, dt \]

October 2, 2020
Discrete linear time-invariant system

Note: discrete time systems is not a focus in this class, but this is useful to know more broadly

\[
\begin{align*}
    x_{k+1} &= ax_k + bu_k \\
    y_k &= cx_k + du_k \\
    x(0) &= x_0,
\end{align*}
\]

where \( x_k \in \mathbb{R}, u_k \in \mathbb{R}, y_k \in \mathbb{R} \), and \( a, b, c, d \in \mathbb{R} \).

**Exercise.** Derive that the solution to the differential equation in \( x \) given by (4) is

\[
x_k = a^k x_0 + \sum_{i=0}^{k-1} a^{k-i-1} bu_i
\]

**Problem 5.** Show that the the system in (4) is a linear and b) time-invariant

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**Time Invar?**

**Req 1:** \( U + Y \) closed under \( T_k \) \( \forall k \)

**Req 2:** \( \text{WTS } \mathcal{P}(T_k \tau, x_0, u) = \mathcal{P}(k_0 + \tau, k_0 + \tau, x_0, T_k(u)) \)
LHS: \( ca^{k-1} x_0 + \sum_{\mathbf{r} \in \mathbf{k}_0} a^{r-1} b(r) + dM(k) \)

RHS: \( c a^{k-1} x_0 + \sum_{\mathbf{r} \in \mathbf{k}_0} a^{r-1} b(r) + dM(k) \)