Problem 1. Note that from the problem description, t is in some interval, where this interval has length less than or equal to T. Thus \( t - t_0 \leq T \). Now, fix \( t \in [t_0, t_0 + T] \)

\[
\|x(t) - z(t)\| = \| \int_{t_0}^{t} p(x(s), s) - p(z(s), s) - f(s)ds - \delta x_0 \|
\]

\[
\leq \| \int_{t_0}^{t} p(x(s), s) - p(z(s), s)ds \| + \| \int_{t_0}^{t} f(s)ds \| + \| \delta x_0 \|
\]

\[
\leq \int_{t_0}^{t} |p(x(s), s) - p(z(s), s)| ds + \epsilon_1(t - t_0) + \epsilon_0
\]

Because \( p(\cdot, \cdot) \) obeys fundamental theorem, apply Lipschitz:

\[
\leq \int_{t_0}^{t} k(s) |x(s) - z(s)| ds + \epsilon_1(t - t_0) + \epsilon_0
\]

\[
\leq \bar{K} \int_{t_0}^{t} |x(s) - z(s)| ds + \epsilon_1(T) + \epsilon_0, \text{ where } \bar{K} = \sup_{s \in [t_0, t_0 + T]} k(s)
\]

Apply Bellman-Gronwall lemma:

\[
\|x(t) - z(t)\| \leq (\epsilon_1(T) + \epsilon_0) * \exp(\bar{K}(t - t_0)) \leq (\epsilon_1(T) + \epsilon_0) * \exp(\bar{K}T)
\]

Intuitively, we can see that this bound is based on an offset from the perturbed initial condition (\( \epsilon_0 \)) plus an offset from the perturbed dynamics over time (\( \epsilon_1(t - t_0) \)), all propagated through time (\( \exp(\bar{K}(t - t_0)) \))

Problem 2. In order to find the state transition function, we must derive the mapping from the initial condition \( x_0 \) to some future time \( i \geq 0 \). Let’s examine how the state evolves starting from \( i = 0 \):

\[
\begin{align*}
x[1] &= ax[0] = ax_0 \\
x[2] &= ax[1] = a^2x_0 \\
x[3] &= ax[2] = a^3x_0 \\
&\vdots \\
x[i] &= ax[i - 1] = a^ix_0
\end{align*}
\]

The state transition function from time \( t_0 = 0 \) to \( t = i \) is given by:

\[
s(i, 0, x[0]) = s(i, 0, x_0) = a^ix_0
\]

In order to determine if the system is linear, we need to examine the response function (which is the composition of the read-out map and the state transition function). It is clear that the state space and the output space are vector spaces. Furthermore, the output is \( y[i] = x[i] = a^ix_0 \). Therefore, the response function is:

\[
\rho(i, 0, x_0) = s(i, 0, x_0) = a^ix_0
\]
To test for linearity, we choose any $\alpha, \beta \in \mathbb{R}$ and any two initial conditions $x_{01}, x_{02} \in \mathbb{R}^n$, then:

$$
\rho(i, 0, \alpha x_{01} + \beta x_{02}) = a^i(\alpha x_{01} + \beta x_{02})
= \alpha a^i x_{01} + \beta a^i x_{02}
= \alpha \rho(i, 0, x_{01}) + \beta \rho(i, 0, x_{02})
$$

Thus, the dynamical system is linear.

**Problem 3.** Recall that a dynamical system is said to be time-invariant if:

1. $U$ is closed under the shift operator $T_r, \forall r$ and
2. $\forall t_0, \forall x_0 \in X, \forall u \in U, r \in T$, and $t_1 \geq t_0$, that $\rho(t_1, t_0 x_0, u) = \rho(t_1 + r, t_0 + r, x_0, T_i(u))$

Recall the shift operator applied to the control input $T_i u(t) : U \rightarrow U$ is defined as $T_i(u(t)) = u(t - i)$ and shifts time by $i$ units, so (1) holds. For (2), let $y(t; u)$ denote the system output under the input $u$ (this will define our response map $\rho(t, u(\cdot))$). We want to show that $y(t + i; T_i(u)) = y(t; u)$ to show time-invariance. Then:

$$
y(t + i; T_i(u)) = \int_{-\infty}^{t+i} e^{-(t+i-\tau)} T_i(u(\tau)) d\tau
= \int_{-\infty}^{t+i} e^{-(t+i-\tau)} u(\tau - i) d\tau
= \int_{-\infty}^{t} e^{-(t-z)} u(z) dz \quad \text{(Change of variables where } z = \tau - i) 
= y(t; u)
$$

This equality shows that the system is time-invariant.