Problem 1: Discrete-time LQR.

Consider the following optimal control problem where we are interested in controlling the output instead of state:

\[
\min_U \sum_{\tau=0}^{N-1} (y_\tau^T Q y_\tau + u_\tau^T R u_\tau)
\]

subject to

\[
\begin{align*}
x_{t+1} &= A x_t + B u_t, \quad t \in \{0, 1, \ldots, N - 1\} \\
y_t &= C x_t, \\
x_0 &= x_{\text{init}}.
\end{align*}
\]

Here, \(U\) is the sequence of control inputs as in lecture. Find an LQR-like sequence of matrix updates that computes the optimal cost-to-go at all times and the optimal feedback controller at all times.

Problem 2: Discrete-time LQR.

In the above problem, suppose the system matrices are given by:

\[
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},
\]

and the cost matrices are given by \(Q = Q_f = \rho_Q I\) and \(R = \rho_R I\). Let \(x_0 = (1, 0)\) and \(N = 20\). Explain how the output, control and cost-to-go change under the optimal feedback when (i) \(\rho_Q = 1, \rho_R = 1\), (ii) \(\rho_Q = 10^3, \rho_R = 1\) and (i i) \(\rho_Q = 1, \rho_R = 10^3\). You can use MATLAB, python, or whatever you like to solve the problem.

Problem 3: Discrete-time LQR.

Suppose that we would like the system to track a reference state trajectory. Derive the optimal LQR control policy and the cost-to-go function for the reference trajectory problem for LTI systems when the reference trajectory \((x^*, u^*)\) is not dynamically feasible, meaning that: \(x^*_{t+1} \neq f(x^*_t, u^*_t)\).