EE222 - Spring'16 - Lecture 1 Notes[1]

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Nonlinear Systems

$$\dot{x} = Ax + Bu \quad \longrightarrow \quad \dot{x} = f(x, u) \tag{1}$$

Analysis:

We use the shorthand notation $\dot{x} = f(x)$ for $\frac{d}{dt}x(t) = f(x(t))$.

$\dot{x} = f(x)$	$f : \mathbb{R}^n \to \mathbb{R}^n$	time-invariant (autonomous)
$\dot{x} = f(t, x)$	$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$	time-varying (non-autonomous)

Design:

$$\dot{x} = f(x, u) \quad u \text{ to be designed as a function of } x.$$

Equilibria

$x = x^*$ is an equilibrium for $\dot{x} = f(x)$ if $f(x^*) = 0$.

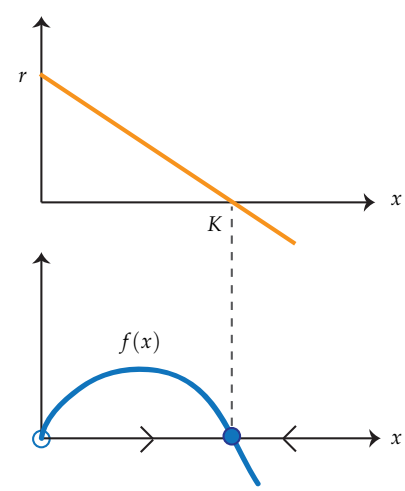
Example: Linear system $\dot{x} = Ax$.

If A is nonsingular, $x^* = 0$ is the unique equilibrium.

If A is singular, the nullspace defines a continuum of equilibria.

Example: *Logistic growth model* in population dynamics

$$\dot{x} = f(x) = \underbrace{r\left(1 - \frac{x}{K}\right)}_{\text{growth rate}} x, \qquad r > 0, \;\; K > 0 \tag{2}$$



$x > 0$ denotes the population and K is called the carrying capacity.

For systems with a scalar state variable $x \in \mathbb{R}$, stability can be determined from the sign of $f(x)$ around the equilibrium. In this example $f(x) > 0$ for $x \in (0, K)$, and $f(x) < 0$ for $x > K$; therefore

$$\begin{array}{ll} x = 0 & \text{unstable equilibrium} \\ x = K & \text{asymptotically stable.} \end{array}$$

Linearization

Local stability properties of x^* can be determined by linearizing the vector field $f(x)$ at x^*:

$$f(x^* + \tilde{x}) = \underbrace{f(x^*)}_{=\,0} + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x=x^*}}_{\triangleq\, A} \tilde{x} + \text{higher order terms} \tag{3}$$

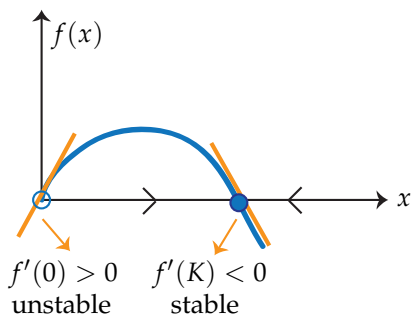
Thus, the linearized model is:

$$\dot{\tilde{x}} = A\tilde{x}. \tag{4}$$

If $\Re\lambda_i(A) < 0$ for each eigenvalue λ_i of A, then x^* is asymp. stable.

If $\Re\lambda_i(A) > 0$ for some eigenvalue λ_i of A, then x^* is unstable.

Example: Logistic growth model above:



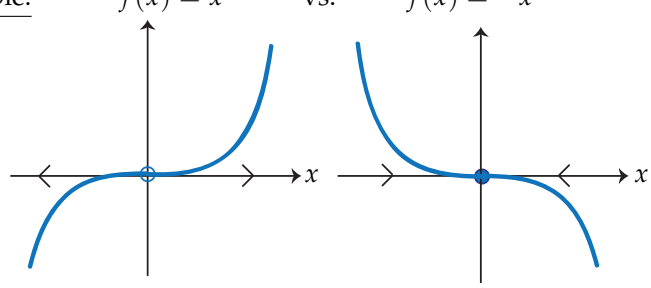
Caveats:

1. Only local properties can be determined from the linearization.

   Example: The logistic growth model linearized at $x = 0$ ($\dot{x} = rx$) would incorrectly predict unbounded growth of $x(t)$. In reality, $x(t) \to K$.

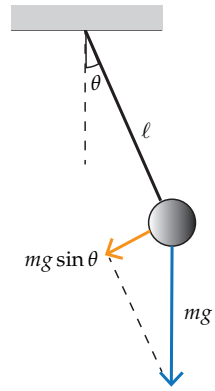
2. If $\Re\lambda_i(A) \leq 0$ with equality for some i, then linearization is inconclusive as a stability test. Higher order terms determine stability.

   Example: $f(x) = x^3$ vs. $f(x) = -x^3$



$f'(0) = 0$ in each case, but one is stable and the other is unstable.

Second order example: Pendulum



$$\ell m\ddot{\theta} = -k\ell\dot{\theta} - mg\sin\theta \tag{5}$$

Define $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$. State space: $S^1 \times \mathbb{R}$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_2 - \frac{g}{\ell}\sin x_1 \end{aligned} \tag{6}$$

Equilibria: $(0,0)$ and $(\pi,0)$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x_1 & -\frac{k}{\ell} \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{k}{\ell} \end{bmatrix} & \text{(stable) at } x_1 = 0 \\ \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{k}{\ell} \end{bmatrix} & \text{(unstable) at } x_1 = \pi \end{cases}$$

Phase portrait: plot of $x_1(t)$ vs. $x_2(t)$ for 2nd order systems

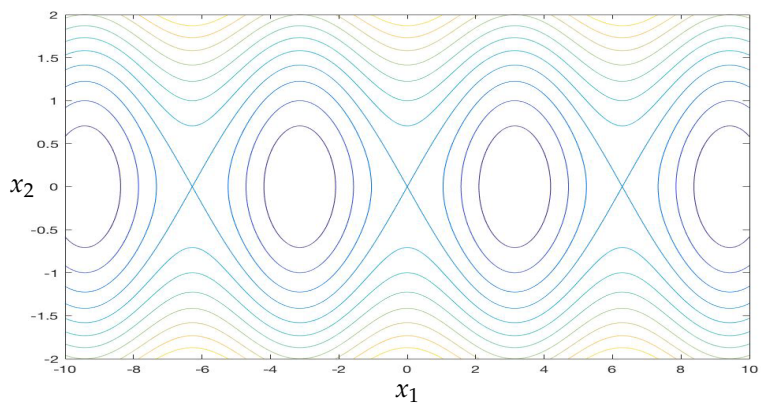


Figure 1: Phase portrait of the pendulum for the undamped case $k = 0$.

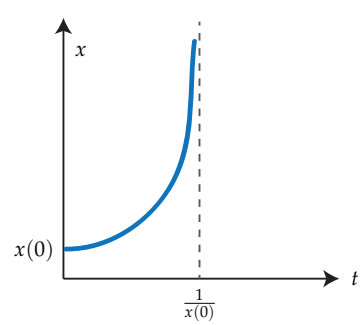
Essentially Nonlinear Phenomena

1. Finite Escape Time

   Example: $\dot{x} = x^2$

$$\begin{aligned} &\frac{d}{dt}x^{-1} = -x^{-2}\dot{x} = -1 \\ \Rightarrow\quad &\frac{1}{x(t)} - \frac{1}{x(0)} = -t \\ \Rightarrow\quad &x(t) = \frac{1}{\frac{1}{x(0)} - t} \end{aligned} \tag{7}$$

   For linear systems, $x(t) \to \infty$ cannot happen in finite time.



2. Multiple Isolated Equilibria

   Linear systems: either unique equilibrium or a continuum

   Pendulum: two isolated equilibria (one stable, one unstable)

   "Multi-stable" systems: two or more stable equilibria

   Example: bistable switch

$$\begin{aligned} \dot{x}_1 &= -ax_1 + x_2 \qquad x_1 : \text{ concentration of protein} \\ \dot{x}_2 &= \frac{x_1^2}{1+x_1^2} - bx_2 \qquad x_2 : \text{ concentration of mRNA} \end{aligned} \tag{8}$$

   $a > 0$, $b > 0$ are constants. State space: $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$.

   This model describes a positive feedback where the protein encoded by a gene stimulates more transcription via the term $\frac{x_1^2}{1+x_1^2}$.

   Single equilibrium at the origin when $ab > 0.5$. If $ab < 0.5$, the line where $\dot{x}_1 = 0$ intersects the sigmoidal curve where $\dot{x}_2 = 0$ at two other points, giving rise to a total of three equilibria: