EE222 - Spring'16 - Lecture 2 Notes[1]

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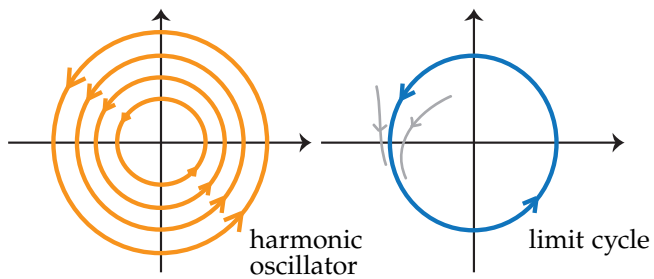
Essentially Nonlinear Phenomena Continued

1. Finite escape time
2. Multiple isolated equilibria
3. Limit cycles: Linear oscillators exhibit a continuum of periodic orbits; *e.g.*, every circle is a periodic orbit for $\dot{x} = Ax$ where

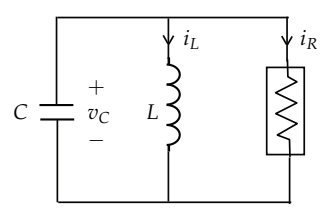
$$A = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} \qquad (\lambda_{1,2} = \mp j\beta).$$

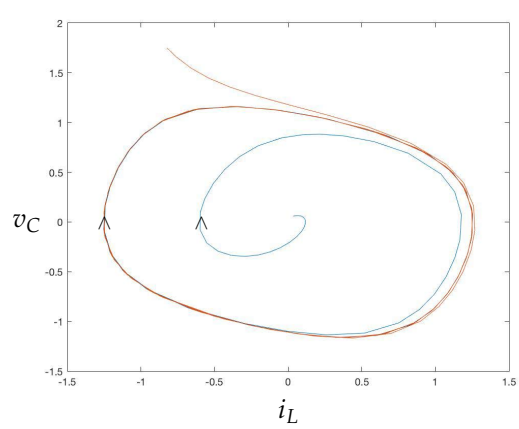
In contrast, a limit cycle is an isolated periodic orbit and can occur only in nonlinear systems.

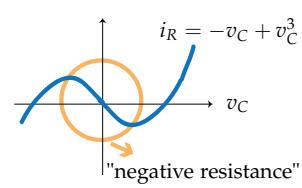


<u>Example:</u> van der Pol oscillator

$$\begin{aligned} C\dot{v}_C &= -i_L + v_C - v_C^3 \\ L\dot{i}_L &= v_C \end{aligned}$$





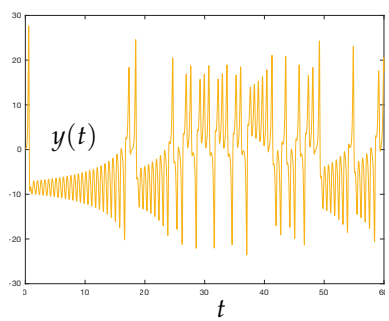


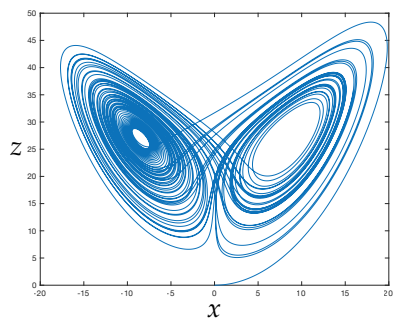
4. Chaos: Irregular oscillations, never exactly repeating.

   Example: Lorenz system (derived by Ed Lorenz in 1963 as a simplified model of convection rolls in the atmosphere):

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz. \end{aligned}$$

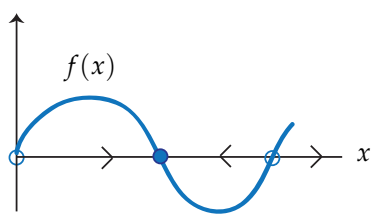
   Chaotic behavior with $\sigma = 10,\ b = 8/3,\ r = 28$:





   - For continuous-time, time-invariant systems, $n \geq 3$ state variables required for chaos.

     $n = 1$: $x(t)$ monotone in t, no oscillations:



     $n = 2$: Poincaré-Bendixson Theorem (to be studied in Lecture 3) guarantees regular behavior.
   - Poincaré-Bendixson does not apply to time-varying systems and $n \geq 2$ is enough for chaos (Homework problem).
   - For discrete-time systems, $n = 1$ is enough (we will see an example in Lecture 5).

Planar (Second Order) Dynamical Systems

Chapter 2 in both Sastry and Khalil

Phase Portraits of Linear Systems: $\dot{x} = Ax$

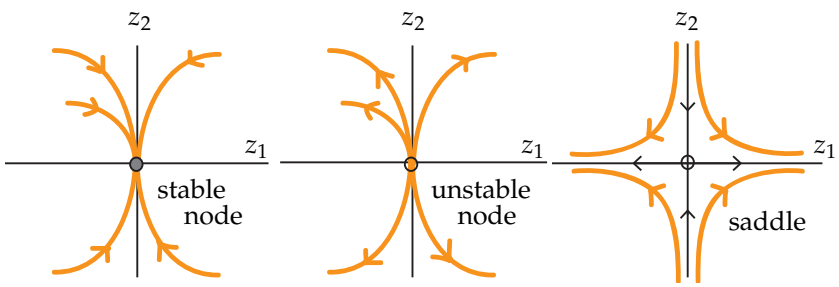
- Distinct real eigenvalues

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

In $z = T^{-1}x$ coordinates:

$$\dot{z}_1 = \lambda_1 z_1, \quad \dot{z}_2 = \lambda_2 z_2.$$

The equilibrium is called a *node* when λ_1 and λ_2 have the same sign (*stable* node when negative and *unstable* when positive). It is called a *saddle point* when λ_1 and λ_2 have opposite signs.

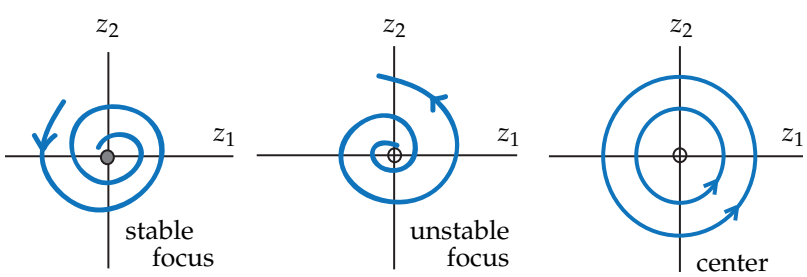


$\lambda_1 < \lambda_2 < 0$ $\qquad$ $\lambda_1 > \lambda_2 > 0$ $\qquad$ $\lambda_2 < 0 < \lambda_1$

- Complex eigenvalues: $\lambda_{1,2} = \alpha \mp j\beta$

$$T^{-1}AT = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$$\begin{aligned} \dot{z}_1 &= \alpha z_1 - \beta z_2 \\ \dot{z}_2 &= \alpha z_2 + \beta z_1 \end{aligned} \quad \rightarrow \quad \text{polar coordinates} \quad \rightarrow \quad \begin{aligned} \dot{r} &= \alpha r \\ \dot{\theta} &= \beta \end{aligned}$$

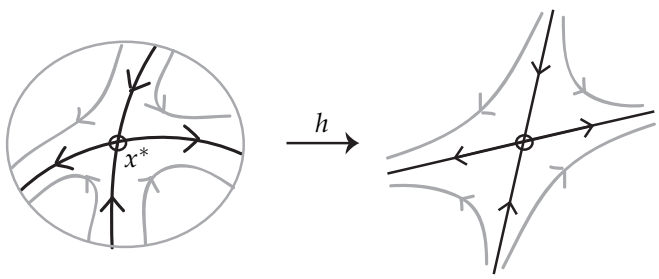


$\alpha < 0$ $\qquad$ $\alpha > 0$ $\qquad$ $\alpha = 0$

Phase Portraits of Nonlinear Systems Near Hyperbolic Equilibria

hyperbolic equilibrium: linearization has no eigenvalues on the imaginary axis

Phase portraits of nonlinear systems near hyperbolic equilibria are qualitatively similar to the phase portraits of their linearization. According to the Hartman-Grobman Theorem (below) a "continuous deformation" maps one phase portrait to the other.



Hartman-Grobman Theorem: If x^* is a hyperbolic equilibrium of $\dot{x} = f(x), x \in \mathbb{R}^n$, then there exists a *homeomorphism*[2] $z = h(x)$ defined in a neighborhood of x^* that maps trajectories of $\dot{x} = f(x)$ to those of $\dot{z} = Az$ where $A \triangleq \left.\frac{\partial f}{\partial x}\right|_{x=x^*}$.

[2] a continuous map with a continuous inverse

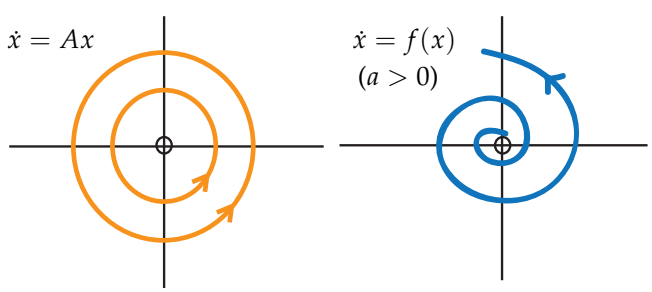
The hyperbolicity condition can't be removed:

Example:

$$\begin{aligned} \dot{x}_1 &= -x_2 + ax_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + ax_2(x_1^1 + x_2^2) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \dot{r} &= ar^3 \\ \dot{\theta} &= 1 \end{aligned}$$

$$x^* = (0,0) \qquad A = \left.\frac{\partial f}{\partial x}\right|_{x=x^*} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

There is no continuous deformation that maps the phase portrait of the linearization to that of the original nonlinear model:

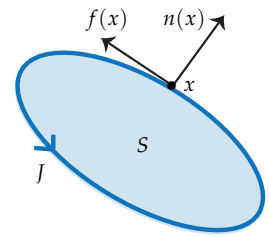


Periodic Orbits in the Plane

Bendixson's Theorem: For a time-invariant planar system

$$\dot{x}_1 = f_1(x_1, x_2) \quad \dot{x}_2 = f_2(x_1, x_2),$$

if $\nabla \cdot f(x) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign in a simply connected region D, then there are no periodic orbits lying entirely in D.



Proof: By contradiction. Suppose a periodic orbit J lies in D. Let S denote the region enclosed by J and $n(x)$ the normal vector to J at x. Then $f(x) \cdot n(x) = 0$ for all $x \in J$. By the Divergence Theorem:

$$\underbrace{\int_J f(x) \cdot n(x) d\ell}_{=0} = \underbrace{\iint_S \nabla \cdot f(x) dx}_{\neq 0}.$$

<u>Example:</u> $\dot{x} = Ax, x \in \mathbb{R}^2$ can have periodic orbits only if

$\text{Trace}(A) = 0$, *e.g.*,

$$A = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}.$$

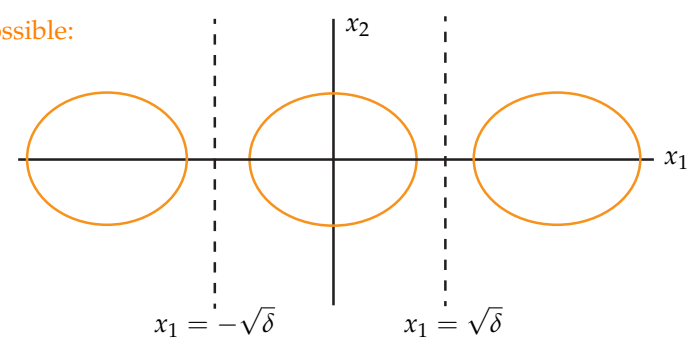
<u>Example:</u>

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\delta x_2 + x_1 - x_1^3 + x_1^2 x_2 \quad \delta > 0 \end{aligned}$$

$$\nabla \cdot f(x) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = x_1^2 - \delta$$

Therefore, no periodic orbit can lie entirely in the region $x_1 \leq -\sqrt{\delta}$ where $\nabla \cdot f(x) \geq 0$, or $-\sqrt{\delta} \leq x_1 \leq \sqrt{\delta}$ where $\nabla \cdot f(x) \leq 0$, or $x_1 \geq \sqrt{\delta}$ where $\nabla \cdot f(x) \geq 0$.

not possible:



possible: