Homework #1

Due: January 24, Wednesday

1. For each of the following first-order differential equations, sketch the direction of motion on the real line, find all equilibrium points, classify their stability, and make a rough sketch of the solution $x(t)$ as a function of time t for various initial conditions:

   a) $\dot{x} = x - x^3$ b) $\dot{x} = e^{-x} \sin x$ c) $\dot{x} = 1 + 0.5 \cos x$ d) $\dot{x} = x^4$.

   Do any of these models exhibit finite escape time?

2. Consider the *logistic growth equation* studied in class, rewritten here as

$$\frac{dx}{(1 - x/K)x} = r dt.$$

   Integrate both sides of this equation to calculate the analytical solution for $x(t)$. Show that $x(t) \to K$ for any $x(0) > 0$.

3. Discuss why the solutions of a *first-order* ordinary differential equation $\dot{x} = f(x)$, where $x \in \mathbb{R}$ and f is time-invariant and continuously differentiable, cannot exhibit oscillations.

4. Duffing's equation, $\ddot{x} + \delta\dot{x} - x + x^3 = \gamma \cos(\omega t)$, exhibits *chaotic* behavior for its parameters in certain ranges. Simulate this equation for $(\delta, \gamma, \omega) = (0.05, 0.4, 1.3)$ and plot the phase portrait (*i.e.*, plot trajectories on the x vs. $\dot{x}$ plane from several initial conditions). You may use a solver such as `ode45` in MATLAB or `odeint` in SciPy.

5. A second order system

$$\dot{x}_1 = f_1(x_1, x_2) \quad \dot{x}_2 = f_2(x_1, x_2)$$

   is said to be "conservative" if there exists an "energy function" $E : \mathbb{R}^2 \mapsto \mathbb{R}$ such that

$$\frac{\partial E(x_1, x_2)}{\partial x_1} f_1(x_1, x_2) + \frac{\partial E(x_1, x_2)}{\partial x_2} f_2(x_1, x_2) = 0$$

   for all x_1, x_2. From the chain rule the equality above implies

$$\frac{d}{dt} E(x_1(t), x_2(t)) = 0,$$

   which means that E is constant along the trajectories of the system. Therefore one can draw a phase portrait by sketching the contours in the x_1 vs. x_2 plane where $E(x_1, x_2)$ is constant.

   a) Show that the pendulum equation with no damping, $\dot{x}_1 = x_2$, $\dot{x}_2 = -\frac{g}{\ell} \sin(x_1)$, is conservative and sketch the phase portrait using the contours of the energy function.

   b) Show that the system $\dot{x}_1 = x_2$, $\dot{x}_2 = x_1 - x_1^3$ is also conservative and sketch the phase portrait using the contours of the energy function.