Homework #10  
Due: April 27, Friday

1. For the system $\dot{x} = f(x)$, $f(0) = 0$, suppose there exists a constant $L$ such that
   \[ |f(x)| \leq L|x| \]  
   for all $x \in \mathbb{R}^n$, where $|\cdot|$ denotes the two norm.
   
   a) Let $W(t) := x(t)^T x(t)$ and show that $|\dot{W}(t)| \leq 2LW(t)$.
   
   b) It follows from (a) that $-2LW(t) \leq \dot{W}(t) \leq 2LW(t)$. Using this fact show that
   \[ |x(0)|e^{-Lt} \leq |x(t)| \leq |x(0)|e^{Lt}. \]

   The lower bound implies that $x(t)$ can’t converge to the origin faster than exponentially; e.g., finite time convergence is not possible. The same conclusion is true when (1) holds locally in a neighborhood of the origin, since trajectories converging to the origin must enter and stay in this neighborhood and, thus, can’t converge faster than exponentially.

2. The dynamics of the translational oscillator with rotating actuator (TORA) depicted below are described by:
   \[
   \begin{align*}
   \dot{x}_1 &= x_2 \\
   \dot{x}_2 &= -x_1 + \epsilon x_2^2 \sin x_3 + \frac{-\epsilon \cos x_3 u}{1 - \epsilon^2 \cos^2 x_3} \\
   \dot{x}_3 &= x_4 \\
   \dot{x}_4 &= \frac{1}{1 - \epsilon^2 \cos^2 x_3} [\epsilon \cos x_3 (x_1 - \epsilon x_2^2 \sin x_3) + u],
   \end{align*}
   \]

   where $x_1$ and $x_2$ are the displacement and the velocity of the platform, $x_3$ and $x_4$ are the angle and angular velocity of the rotor carrying the mass $m$, and $u$ is the control torque applied to the rotor. The parameter $\epsilon < 1$ depends on the eccentricity $e$ and the masses $m$ and $M$.

With $y = x_3$ as the output, determine the relative degree and the zero dynamics. Give a physical interpretation of the zero dynamics.
3. Consider the system

\[
\begin{align*}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= x_4 + x_3 u_1 \\
\dot{x}_3 &= -x_3 + x_4 \\
\dot{x}_4 &= u_2
\end{align*}
\]

a) Let the two outputs be \( y_1 = x_1 \) and \( y_2 = x_2 \), and show that the system is not input/output linearizable; that is, there is no well-defined vector relative degree.

b) In this example it is possible to overcome the obstacle to input/output linearization by setting \( u_1 \) equal to the output of an integrator, to be implemented as part of a dynamic controller:

\[
\begin{align*}
u_1 &= x_5 \\
\dot{x}_5 &= \tilde{u}_1.
\end{align*}
\]

Show that the augmented system has a well-defined vector relative degree with respect to the inputs \( \tilde{u}_1 \) and \( u_2 \).

4. The following system is to be stabilized at the origin:

\[
\begin{align*}
\dot{x}_1 &= x_1 x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u.
\end{align*}
\]

(2)

a) Show that the system is not feedback linearizable.

b) In contrast, backstepping is applicable. Select a virtual control law for the variable \( x_2 \) to stabilize the \( x_1 \) subsystem, and apply one step of backstepping to obtain a virtual control law for \( x_3 \). (No need to apply the second step to design a control law for \( u \).)

5. a) Use your answer to Problem 4(b) to propose a sliding surface and design a sliding mode controller to stabilize the origin. Simulate the closed loop system from several initial conditions and plot the trajectories as functions of time.

b) Now suppose the equation for \( x_3 \) in (2) is replaced by:

\[ \dot{x}_3 = (1 + \theta) u \quad |\theta| \leq 0.5. \]

Modify the sliding mode controller you designed in part (a) to account for the uncertain parameter \( \theta \). (You are not asked to run simulations for this part.)