Homework #10

Due: April 27, Friday

1. For the system $\dot{x} = f(x)$, $f(0) = 0$, suppose there exists a constant L such that

$$|f(x)| \leq L|x| \tag{1}$$

for all $x \in \mathbb{R}^n$, where $|\cdot|$ denotes the two norm.

a) Let $W(t) := x(t)^T x(t)$ and show that $|\dot{W}(t)| \leq 2LW(t)$.

b) It follows from (a) that $-2LW(t) \leq \dot{W}(t) \leq 2LW(t)$. Using this fact show that

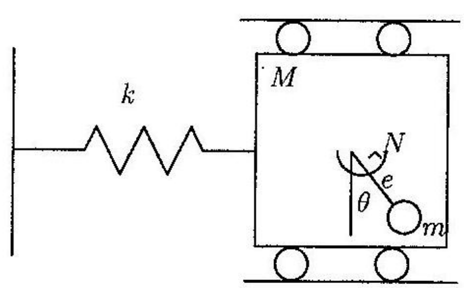
$$|x(0)|e^{-Lt} \leq |x(t)| \leq |x(0)|e^{Lt}.$$

The lower bound implies that $x(t)$ can't converge to the origin faster than exponentially; *e.g.*, finite time convergence is not possible. The same conclusion is true when (1) holds locally in a neighborhood of the origin, since trajectories converging to the origin must enter and stay in this neighborhood and, thus, can't converge faster than exponentially.

2. The dynamics of the *translational oscillator with rotating actuator* (TORA) depicted below are described by:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{-x_1 + \epsilon x_4^2 \sin x_3}{1 - \epsilon^2 \cos^2 x_3} + \frac{-\epsilon \cos x_3}{1 - \epsilon^2 \cos^2 x_3} u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{1}{1 - \epsilon^2 \cos^2 x_3} [\epsilon \cos x_3 (x_1 - \epsilon x_4^2 \sin x_3) + u],
\end{aligned}$$

where x_1 and x_2 are the displacement and the velocity of the platform, x_3 and x_4 are the angle and angular velocity of the rotor carrying the mass m, and u is the control torque applied to the rotor. The parameter $\epsilon < 1$ depends on the eccentricity e and the masses m and M.



With $y = x_3$ as the output, determine the relative degree and the zero dynamics. Give a physical interpretation of the zero dynamics.

3. Consider the system

$$
\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= x_4 + x_3 u_1 \\
\dot{x}_3 &= -x_3 + x_4 \\
\dot{x}_4 &= u_2
\end{aligned}
$$

a) Let the two outputs be $y_1 = x_1$ and $y_2 = x_2$, and show that the system is not input/output linearizable; that is, there is no well-defined vector relative degree.

b) In this example it is possible to overcome the obstacle to input/output linearization by setting u_1 equal to the output of an integrator, to be implemented as part of a dynamic controller:

$$
\begin{aligned}
u_1 &= x_5 \\
\dot{x}_5 &= \tilde{u}_1.
\end{aligned}
$$

Show that the augmented system has a well-defined vector relative degree with respect to the inputs $\tilde{u}_1$ and u_2.

4. The following system is to be stabilized at the origin:

$$
\begin{aligned}
\dot{x}_1 &= x_1 x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u.
\end{aligned}
\tag{2}
$$

a) Show that the system is not feedback linearizable.

b) In contrast, backstepping *is* applicable. Select a virtual control law for the variable x_2 to stabilize the x_1 subsystem, and apply one step of backstepping to obtain a virtual control law for x_3. (No need to apply the second step to design a control law for u.)

5. a) Use your answer to Problem 4(b) to propose a sliding surface and design a sliding mode controller to stabilize the origin. Simulate the closed loop system from several initial conditions and plot the trajectories as functions of time.

b) Now suppose the equation for x_3 in (2) is replaced by:

$$
\dot{x}_3 = (1+\theta)u, \quad |\theta| \leq 0.5.
$$

Modify the sliding mode controller you designed in part (a) to account for the uncertain parameter θ. (You are *not* asked to run simulations for this part.)