Homework #2

Due: January 31, Wednesday

1. The Lotka-Volterra population model for two species competing for resources is

$$\dot{x}_1 = r_1 x_1 \left(\frac{K_1 - x_1 - \beta_{12} x_2}{K_1} \right), \quad \dot{x}_2 = r_2 x_2 \left(\frac{K_2 - x_2 - \beta_{21} x_1}{K_2} \right),$$

where $x_i \geq 0$ is the population, $r_i > 0$ is the growth rate, and $K_i > 0$ is the carrying capacity for species $i = 1, 2$, and $\beta_{12} > 0, \beta_{21} > 0$ account for the competition. Note that the following are equilibrium points: $x_1 = 0, x_2 = 0$ (both species absent), $x_1 = K_1, x_2 = 0$ (species 2 absent, species 1 at its carrying capacity), $x_1 = 0, x_2 = K_2$ (species 1 absent, species 2 at capacity).

a) Under what conditions on the parameters does there exist another equilibrium in the positive quadrant $\mathbb{R}^2_{>0}$?

b) Analyze the stability of each equilibrium, including the one in part (a) when it exists.

2. Consider the following system discussed in Lecture 1:

$$\dot{x}_1 = -a x_1 + x_2, \quad \dot{x}_2 = \frac{x_1^2}{1 + x_1^2} - b x_2, \quad a, b > 0.$$

a) Show that the nonnegative quadrant $\mathbb{R}^2_{\geq 0}$ is positively invariant.

b) Give a condition on the parameters a and b so that $\mathbb{R}^2_{\geq 0}$ contains multiple equilibria. Determine whether each equilibrium is a stable (unstable) node, focus, or a saddle.

c) Simulate the system for a sample choice of parameters satisfying the condition derived in part (b) and plot the phase portrait.

3. Design a second order system with a stable focus and a stable limit cycle. You may use polar coordinates, but the final system equations should be given in the Cartesian coordinates.

4. Prove the following generalization of Bendixson's Criterion, known as the *Bendixson-Dulac Criterion*: For a time-invariant planar system $\dot{x}_1 = f_1(x_1, x_2)$, $\dot{x}_2 = f_2(x_1, x_2)$, if there exists a continuously differentiable function $B : \mathbb{R}^2 \mapsto \mathbb{R}$ such that

$$\nabla \cdot (B(x) f(x)) = \frac{\partial (B(x) f_1(x))}{\partial x_1} + \frac{\partial (B(x) f_2(x))}{\partial x_2}$$

is not identically zero and does not change sign in a simply connected region D, then there are no periodic orbits lying entirely in D.

5. a) Show that Bendixson's Criterion *cannot* be used to argue the absence of periodic orbits in $\mathbb{R}^2_{>0}$ for the two-species competition model in Problem 1.

b) Now use the extended criterion in Problem 4 to show the absence of periodic orbits in $\mathbb{R}^2_{>0}$ for the two-species competition model in Problem 1. (Hint: try $B(x_1, x_2) = 1/(x_1 x_2)$.)