Homework #3

Due: February 7, Wednesday

1. Consider the system

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = u$$

where the control input can take only the values ± 1.

a) Sketch the phase portraits for $u = 1$ and $u = -1$.

b) Superimpose the phase portraits and develop a strategy for switching the control between ± 1 so that any point in the space can be moved to the origin in finite time.

2. Let x_1 denote the number of vehicles on a stretch of highway and let x_2 be the number of vehicles queued to enter this stretch. The so-called *cell transmission model* (CTM) determines the flow into the highway from the expression $\min\{D(x_2), S(x_1)\}$, where $D(x_2)$ is the flow wishing to enter the highway and $S(x_1)$ is flow that the highway can accommodate given its current occupancy. D is called the *demand* function and is typically modeled as

$$D(x) = \min\{c, vx\}, \; c > 0, \, v > 0,$$

as depicted below. S is called the *supply* function and is modeled as

$$S(x) = w(\bar{x} - x), \; w > 0, \; x \in [0, \bar{x}],$$

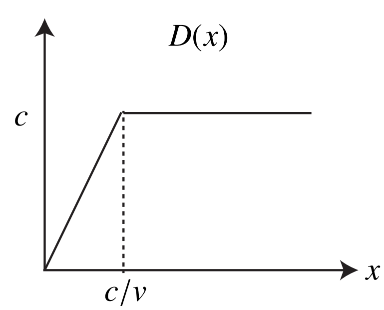
where $\bar{x}$ is the *jam* occupancy. The evolution of x_1 and x_2 are then described by

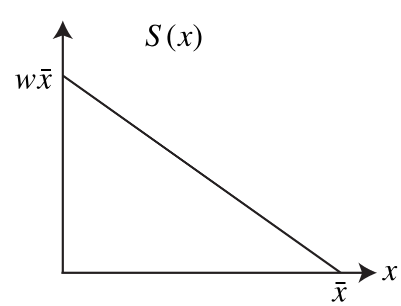
$$\dot{x}_1 = \min\{D(x_2), S(x_1)\} - D(x_1) \quad \dot{x}_2 = -\min\{D(x_2), S(x_1)\} + d$$

where $d > 0$ is the number of vehicles joining the queue per minute and the term $-D(x_1)$ in the first equation is the flow out of this stretch of highway, assuming no congestion downstream.

a) Show that the set $[0, \bar{x}] \times \mathbb{R}_{\geq 0}$ is positively invariant; that is, the trajectories $x_1(t)$, $x_2(t)$ generated by the model can't become negative and $x_1(t)$ can't exceed the jam occupancy.

b) Using the values $c = 80$ vehicles/minute, $v = 1$ minute^{-1}, $w = 1/3$ minute^{-1}, $\bar{x} = 320$ vehicles, find the equilibrium points (if they exist) for the cases $d < c$, $d = c$, and $d > c$.





3. Consider the predator-prey model from Lecture 3 where the parameters a, b, c, d are positive:

$$\dot{x} = (a - by)x \quad \dot{y} = (cx - d)y.$$

   a) Linearize this model at $(x, y) = (d/c, a/b)$ and discuss whether the stability properties and the phase portrait near this equilibrium can be characterized from the linearization.

   b) Now show that the system exhibits periodic orbits centered at $(x, y) = (d/c, a/b)$ by finding a conserved function as in Homework 1, Problem 5, that has a minimum or maximum at $(x, y) = (d/c, a/b)$. (Hint: try a separable function $E(x, y) = F(x) + G(y)$.)

   c) Plot the phase portrait using numerical simulations to confirm the periodic orbits.

4. Make a bifurcation plot for each of the following two models:

   a) $\dot{x} = \mu + x - \ln(1 + x)$

   b) $\dot{x} = \mu x - \ln(1 + x)$.

   Identify the bifurcation values for the parameter μ and the type of bifurcation that occurs.

5. Consider the discrete-time system:

$$x_{n+1} = x_n^2 + \mu.$$

   a) Find and classify all fixed points as a function of μ.

   b) Find the bifurcation point for the parameter μ and classify the bifurcation.

   c) For which values of μ is there a period-two cycle?

   d) Plot cobweb diagrams for several values of μ.