Homework #4

Due: February 14, Wednesday

1. Consider a mass attached to two springs as shown below. The mass is constrained to move in the vertical direction and each spring is linear with constant $k > 0$ and resting length $b > 0$.

   a) Show that the restoring force in the vertical direction is given by

$$2k\left(y - b\frac{y}{\sqrt{y^2+a^2}}\right)$$

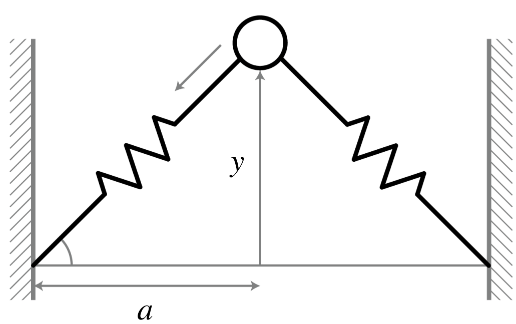
   where y is the displacement and a is the distance indicated in the figure below.

   b) With the state variables $x_1 = y$, $x_2 = \dot{y}$, the resulting equations of motion are

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = -\frac{2k}{m}\left(x_1 - b\frac{x_1}{\sqrt{x_1^2+a^2}}\right) - \frac{d}{m}x_2$$

   where $m > 0$ is the mass and $d > 0$ is the damping coefficient.

   Show that this system undergoes a bifurcation as the ratio $\mu := b/a$ crosses a critical value. Determine the critical value and the type of bifurcation.



2. The *ring oscillator* circuit depicted below consists of a feedback loop of three identical inverters, modeled as

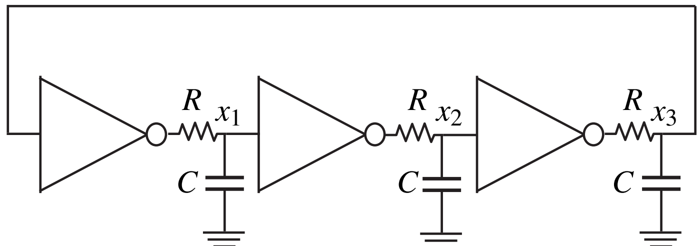
$$\tau\dot{x}_1 = -x_1 + h(x_3), \quad \tau\dot{x}_2 = -x_2 + h(x_1), \quad \tau\dot{x}_3 = -x_3 + h(x_2),$$

   where $\tau = RC > 0$, x_i, $i = 1,2,3$ are the voltages, and the function h representing the inverter characteristics is

$$h(x) = -\alpha\tanh(\beta x) \quad \alpha > 0, \beta > 0.$$

   a) Linearize the model at the origin and show that a complex conjugate pair of eigenvalues crosses the imaginary axis at a critical value of $\mu := \alpha\beta$ to be determined.

b) Let $\tau = 1$ and simulate the system from from $t = 0$ to $t = 100$ for several initial conditions and for values of μ below and above the critical value. You should observe limit cycle oscillations beyond the critical value, indicative of a Hopf bifurcation.



3. Consider the following system studied in Lecture 1:

$$\frac{dx_1}{dt} = -ax_1 + x_2, \quad \frac{dx_2}{dt} = \frac{x_1^2}{1+x_1^2} - bx_2, \quad a, b > 0.$$

a) Take $a = 1$ and $b = 1/2$, and define the shifted state variables $\tilde{x}_1 = x_1 - 1$, $\tilde{x}_2 = x_2 - 1$ so that the equilibrium $x = (1,1)$ is now $\tilde{x} = (0,0)$. Rewrite the state equations above in terms of $\tilde{x}_1$ and $\tilde{x}_2$, eliminating all x_1 and x_2 terms.

b) Linearize the model found in part (a) at $\tilde{x} = (0,0)$ and show that one eigenvalue is at zero.

c) Find variables y and z to bring the system to the form discussed in Lecture 5, equation (2). You can use the eigenvectors of the linearization in part (b) to do so. Next rewrite the state equations in terms of y and z, eliminating $\tilde{x}_1$ and $\tilde{x}_2$ terms.

d) (Optional) Use the center manifold theory to find out if $(y,z) = (0,0)$ is stable or unstable.

4. (Nondimensionalization) Consider the dynamical model:

$$\frac{d}{d\tau}X_1 = -\alpha X_1 + \beta X_2, \quad \frac{d}{d\tau}X_2 = \frac{\gamma X_1^2}{\delta + X_1^2} - \nu X_2,$$

where τ denotes time and X_1, X_2 denote concentrations of two chemical species. Show that this model can be brought to the form in Problem 3 above by defining the scaled variables:

$$x_1 = \frac{X_1}{\bar{X}_1}, \quad x_2 = \frac{X_2}{\bar{X}_2}, \quad t = \frac{\tau}{\bar{\tau}},$$

and selecting the constants $\bar{X}_1$, $\bar{X}_2$, $\bar{\tau}$ appropriately. Express the new parameters a, b, in terms of the old ones α, β, γ, δ, ν.

A procedure like this is useful in modeling physical systems, as it allows one to obtain dimensionless state variables and parameters that do not depend on the choice of units, and to reduce the number of relevant parameters in the model.

5. Sketch each function below and state whether it is continuous, continuously differentiable, locally Lipschitz, globally Lipschitz:

a) $x|x|$ b) $x + |x|$ c) $x + \text{sign}(x)$ d) $x^2 + |x|$ e) $\min\{1, x\}$.