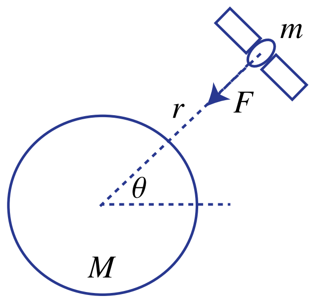
Homework #5

Due: February 21, Wednesday

1. (Orbits of Satellites) The gravitational attraction force experienced by a satellite is

$$F = \frac{GMm}{r^2}$$

where M and m are the masses of the earth and satellite, r is the distance of the satellite from the center of the earth, and G is the universal gravitational constant.



a) Let θ be the angle with respect to a fixed axis and show that the equations of motion are

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}, \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0, \tag{1}$$

where the left hand sides represent the acceleration in the radial and transverse directions.

b) (Circular orbits) Show that the equations above admit a solution of the form $r(t) = \bar{r}$, $\dot{\theta}(t) = \bar{\omega}$, where $\bar{r}$ and $\bar{\omega}$ are constants satisfying $\bar{r}^3\bar{\omega}^2 = GM$.

c) (Conservation of angular momentum) Show $\frac{d}{dt}\left(r(t)^2\dot{\theta}(t)\right) = 0$, *i.e.*, $r(t)^2\dot{\theta}(t) = c := r(0)^2\dot{\theta}(0)$.

You will next identify orbits of the form $r = \frac{1}{u(\theta)}$ where $u(\cdot)$ is a function to be characterized:

d) Show that if $r(t)$ and $\theta(t)$ are to satisfy $r = \frac{1}{u(\theta)}$ for all t, then

$$\dot{r}(t) = -cu'(\theta(t)) \quad \text{and} \quad \ddot{r}(t) = -c^2u^2(\theta(t))u''(\theta(t)),$$

where c is the constant defined in part (c), $u'(\theta) = \frac{du(\theta)}{d\theta}$ and $u''(\theta) = \frac{d^2u(\theta)}{d\theta^2}$.

e) Using the left equation in (1) show that $u(\theta)$ must satisfy

$$u''(\theta) + u(\theta) = \frac{GM}{c^2}. \tag{2}$$

f) The general solution of (2) is $u(\theta) = A\cos(\theta - \phi) + \frac{GM}{c^2}$; thus, the orbits are of the form

$$r = \frac{1}{u(\theta)} = \frac{1}{A\cos(\theta-\phi) + \frac{GM}{c^2}}, \tag{3}$$

where A and ϕ depend on initial conditions. Using (3) and the $\dot{r}$ expression in part (d) with $u'(\theta) = -A\sin(\theta - \phi)$, determine $A \geq 0$[1] as a function of $\dot{\theta}(0)$, $r(0)$, $\dot{r}(0)$, G, and M.

[1] There is no loss of generality in taking $A \geq 0$ because, if (A, ϕ) is a solution, so is $(-A, \phi + \pi)$.

g) In polar coordinates equation (3) defines a circle when $A = 0$, an ellipse when $0 < A < \frac{GM}{c^2}$, a parabola when $A = \frac{GM}{c^2}$, and a hyperbola when $A > \frac{GM}{c^2}$.

Take $M = 5.9722 \cdot 10^{24}$ kg, $G = 6.674 \cdot 10^{-20}$ km^3/kg/s^2, $\theta(0) = 0$, $\dot{r}(0) = 0$, and $r(0) = 6778$ km, which means 400 km from the surface of the earth. Select $\dot{\theta}(0)$ such that $A = 0$ and two others values such that A is in the interval $0 < A < \frac{GM}{c^2}$, simulate the model for each, and plot $r\cos(\theta)$ vs. $r\sin(\theta)$. Choose the time interval for simulation long enough to traverse the resulting orbit at least once (typically in the order of hours, longer as A gets closer to $\frac{GM}{c^2}$ and the eccentricity of the ellipse increases). You may also have to lower the tolerances in your ode solver, e.g., `options = odeset('RelTol', 1e-6, 'AbsTol', 1e-9)` in Matlab.

2. Let X be the space of real-valued continuous functions on $[0,1]$ with norm: $|x|_X = \int_0^1 |x(t)|dt$. This space is different from $C[0,1]$ discussed in Lecture 6, because the norm is different.

   a) Show that the norm $|x|_X = \int_0^1 |x(t)|dt$ satisfies the three properties discussed in Lecture 6.

   b) Show that the sequence below is Cauchy but its limit is not an element of X, i.e., X is not a Banach space.

$$x_n(t) = \begin{cases} 0 & \text{for } 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ nt - \frac{n}{2} + 1 & \text{for } \frac{1}{2} - \frac{1}{n} \le t \le \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} \le t. \end{cases}$$

   c) Unlike X, $C[0,1]$ is a Banach space. Discuss why your argument in part (b) is no longer valid when you use the norm $|x|_C = \max_{t\in[0,1]} |x(t)|$.

3. Explain in your words why Fixed Point theorems are relevant to the existence of solutions to differential equations.

4. Consider the following model of a DC motor under constant field excitation:

$$J\frac{d\omega}{dt} = Ki - T, \quad L\frac{di}{dt} = -K\omega - Ri + U,$$

   where i, U, R, L are the armature current, voltage, resistance, and inductance, J is the moment of inertia, ω is the angular speed, T is the load torque, and K is a positive constant. The terms Ki and $K\omega$ are, respectively, the motor torque and the back electromotive force.

   a) Calculate the equilibrium point as a function of the parameters and show that it is stable.

   b) Find the equilibrium when $T = 14.3$ Nm, $U = 246.8$ V, $R = 7.56\ \Omega$, $L = 0.055$ H, $K = 4.23$ Vs/rad, and $J = 0.136$ kg m^2.

   c) Simulate the model from $t = 0$ to $t = 1$ s, starting at the equilibrium found in part (b), but with a 1 V step increase to $U = 247.8$ V. Plot the resulting transient response of $i(t)$ and $\omega(t)$.

   d) Plot the sensitivities of the trajectories in part (c) with respect to the <u>logarithm</u> of the parameters K, J, R, L, T, V, again from $t = 0$ to $t = 1$ s. Which two parameters have the least effect on the trajectories?

   e) You should see that the sensitivity of $i(t)$ with respect to all parameters except for K and T vanishes in steady state. Explain why.