Homework #6

Due: February 28, Wednesday

1. Using numerical simulations plot the phase portrait of the second order system

$$\dot{x}_1 = \frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)[1 + (x_1^2 + x_2^2)^2]}, \quad \dot{x}_2 = \frac{x_2^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)[1 + (x_1^2 + x_2^2)^2]}.$$

   Do the solutions converge to the origin? Does the origin appear to be stable?

2. The Bloch Equation, commonly used in nuclear magnetic resonance (NMR) and magnetic resonance imaging (MRI), is a model for the magnetization experienced by nuclei of atoms when exposed to a magnetic field. Assuming a constant magnetic field, B_0, along the longitudinal axis z, the x, y, and z components of magnetization evolve according to

$$\dot{M}_x = -\frac{1}{T_2} M_x + \gamma B_0 M_y \quad \dot{M}_y = -\frac{1}{T_2} M_y - \gamma B_0 M_x \quad \dot{M}_z = -\frac{1}{T_1}(M_z - M_0),$$

   where γ is the gyromagnetic ratio that depends on the chemical structure of the atom. Trajectories generated by this model precess in the transverse xy plane, decaying with time constant T_2. Simultaneously the longitudinal component returns to the equilibrium magnetization M_0 with time constant T_1; see `https://www.youtube.com/watch?v=ygwESjbb3rQ`.

   Find a Lyapunov function that shows the stability of the equilibrium $(M_x, M_y, M_z) = (0, 0, M_0)$.

3. Consider a system described by the second order differential equation:

$$\ddot{y} + h(y)\dot{y} + g(y) = 0$$

   where $yg(y) > 0$ for all $y \neq 0$ and $h(y) > 0$ for all y.

   a) Using $x_1 = y$ and $x_2 = \dot{y}$, write state equations and show that the origin is the unique equilibrium.

   b) Using an appropriate Lyapunov function show that the origin is asymptotically stable. What further conditions do you need to conclude *global* asymptotic stability?

4. *Young's Inequality* states that, for any $p > 1, q > 1$ satisfying $(p-1)(q-1) = 1$,

$$xy \leq \frac{\lambda^p}{p}|x|^p + \frac{1}{q\lambda^q}|y|^q \tag{1}$$

   where $\lambda > 0$ can be selected arbitrarily. Use (1) with appropriate choices of p and q to upper bound $x^m y^n$ by a weighted sum of: (a) $|x|^{2m}$ and $|y|^{2n}$, (b) $|x|^{m+n}$ and $|y|^{m+n}$.

5. Find appropriate Lyapunov functions to verify global asymptotic stability of the origin for:

   a) $\dot{x} = -x + y^3 \quad \dot{y} = -y$,

   b) $\dot{x} = -x^3 + y^3 \quad \dot{y} = -y$.

   Hint: Problem 4 may help in finding upper bounds on the cross terms of x and y.