Homework #7

Due: March 7, Wednesday

1. Consider the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -g(k_1 x_1 + k_2 x_2), \quad k_1, k_2 > 0,$$

where the nonlinearity $g(\cdot)$ is such that

$$g(y)y > 0 \ \forall y \neq 0 \quad \text{and} \quad \lim_{|y|\to\infty} \int_0^y g(z)dz = +\infty.$$

a) Using an appropriate Lyapunov function, show that the equilibrium $x = 0$ is globally asymptotically stable.

b) Show that the saturation function $\text{sat}(y) = \text{sign(y)}\min\{1, |\text{y}|\}$ satisfies the assumptions for $g(\cdot)$ above. What is the exact form of your Lyapunov function in this case?

c) Parts (a) and (b) imply that a double integrator with a saturating actuator

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \text{sat}(u)$$

can be stabilized with the state-feedback controller $u = -k_1 x_1 - k_2 x_2$. Design k_1 and k_2 to place the eigenvalues of the linearization at $-1 \pm j$, and simulate the resulting closed-loop system both with and without saturation. Plot $x_1(t)$ and $x_2(t)$ as functions of time (rather than phase portraits) and compare the trajectories with and without saturation.

2. Show that, if the equilibrium $x = 0$ of $\dot{x} = Ax$ is stable, then there exists a matrix $P = P^T > 0$ satisfying:

$$A^T P + PA \leq 0. \tag{1}$$

Hint: Stability implies that the matrix A can be brought (upon a similarity transformation) to the form

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & A_- \end{bmatrix}$$

where A_- has eigenvalues with strictly negative real parts and A_0 has all eigenvalues on the imaginary axis, with Jordan blocks of order one. Thus we can choose A_0 to be a block diagonal matrix with either 0-blocks (corresponding to eigenvalues at 0), or blocks of the form

$$\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

(corresponding to eigenvalues $\pm i\omega$).

3. In this problem you will study the linear system $\dot{x} = Ax$, and show that for any constant α which is greater than the real parts of all eigenvalues of A, there exists a $\kappa > 0$ such that

$$|x(t)| \leq \kappa e^{\alpha t}|x(0)|. \tag{2}$$

a) Show that if $\alpha > \text{Re}\{\lambda_i\}$ for all eigenvalues λ_i of A, then there exists a matrix $P = P^T > 0$ such that

$$A^T P + PA \leq 2\alpha P. \tag{3}$$

Hint: Note that $A - \alpha I$ is Hurwitz.

b) Using (3), show that the function $V(x) = x^T Px$ satisfies

$$V(x(t)) \leq V(x(0))e^{2\alpha t}. \tag{4}$$

c) Show that (2) follows from (4), and give a formula for κ in terms of $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, the minimum and maximum eigenvalues of P.

4. For a discrete-time system $x_{k+1} = f(x_k)$, the origin $x = 0$ is asymptotically stable if there exists a positive definite Lyapunov function $V(x)$ such that

$$V(x_{k+1}) - V(x_k) = -W(x_k)$$

for some positive definite $W(x)$. Using $V(x) = x^T Px$ for the linear time-invariant system

$$x_{k+1} = Ax_k,$$

state and prove a theorem that is analogous to the continuous-time result that relates the existence of $P = P^T > 0$ satisfying $A^T P + PA = -Q < 0$ to the Hurwitz property of A.

5. Consider the reference model:

$$\dot{y}_m = -ay_m + r(t), \quad a > 0,$$

and the plant:

$$\dot{y} = a^* y + b^* u, \quad b^* \neq 0.$$

a) Show that a controller of the form:

$$u = \theta_1 y + \theta_2 r(t)$$

with an appropriate choice of gains θ_1^* and θ_2^*, drives the tracking error $e := y - y_m$ to zero asymptotically.

b) Now suppose a^* and b^* are unknown parameters, but the sign of b^* is known. Show that the adaptive implementation of the controller above achieves tracking when the gains are updated according to the rule:

$$\dot{\theta}_1 = -\text{sgn}(b^*)\gamma_1 ye, \quad \dot{\theta}_2 = -\text{sgn}(b^*)\gamma_2 re, \tag{5}$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$.

c) Simulate this controller for several choices of $r(t)$ and observe whether or not $\theta_1(t) \to \theta_1^*$ and $\theta_2(t) \to \theta_2^*$ as $t \to \infty$ for each choice.