Homework #8

Due: March 21, Wednesday

1. Consider the linear system $\dot{x} = Ax$ where A is a 3×3 skew-symmetric matrix:

$$A = -A^T. \tag{1}$$

a) Show that $\|x(t)\|_2$, the Euclidean norm of $x(t)$, remains constant along the solutions; that is, the trajectories evolve on spheres.

b) Show that the trajectories are in fact fixed circles in $\mathbb{R}^3$. (*Hint:* It follows from (1) that A is singular, because $\det(A) = \det(-A^T) = \det(-A) = (-1)^3 \det(A) = -\det(A)$. Thus, there exists a vector $v \neq 0$ such that $v^T A = 0$.)

2. For the linear time-varying system

$$\dot{x} = A(t)x$$

Liouville's Theorem states that the determinant of the state-transition matrix,

$$W(t) = \det\{\Phi(t, 0)\},$$

satisfies the equation

$$\dot{W}(t) = a(t)W(t) \quad W(0) = 1,$$

where $a(t) := \mathrm{Trace}\{A(t)\}$.

a) What can you say about the asymptotic stability of $x = 0$ if you determine that $W(t)$ does <u>not</u> converge to zero? Can you conclude asymptotic stability if it does converge to zero?

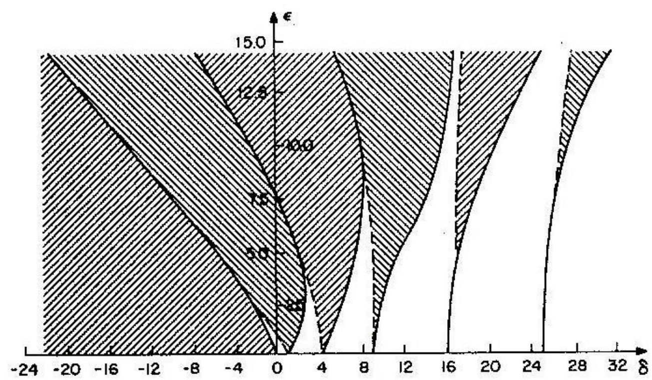
b) The Mathieu Equation,

$$\ddot{x} + (\delta + \epsilon \cos 2t)x = 0,$$

governs the forced motion of a swing, stability of ships and columns, the dynamics of electrons in Penning traps, the behavior of parametric amplifiers that use electronic or superconducting devices, and resonances in microelectromechanical systems.

Use the conclusions of part (a) to show that the equilibrium $x = 0$ is <u>not</u> asymptotically stable.

c) The equilibrium has been shown to be unstable for values of the parameters δ and ϵ in the shaded regions of the figure below, and to be stable in the remaining regions. Choose one set of parameters from the stable region, and another set from the unstable region, and simulate the solutions of $x(t)$.



3. Consider the system
$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f(0) = 0,$$
and suppose the Jacobian matrix $J(x) := \frac{\partial f(x)}{\partial x}$ satisfies
$$PJ(x) + J(x)^T P \leq -I \qquad \forall x \in \mathbb{R}^n$$
for some $P = P^T > 0$.

a) Using the representation $f(x) = \int_0^1 J(sx)x ds$, obtained from the Mean Value Theorem, show that
$$x^T P f(x) + f(x)^T P x \leq -x^T x \qquad \forall x \in \mathbb{R}^n.$$
Note that this inequality implies global asymptotic stability of the origin from the Lyapunov function $x^T P x$.

b) An alternative Lyapunov function is $V(x) = f(x)^T P f(x)$. Show that this function is positive definite and yields a negative definite $\dot{V}(x)$. *Optional:* Show that V is also radially unbounbded.

4. The following axial compressor model, used in jet engine control studies, captures the surge instability between mass flow and pressure rise:
$$\begin{aligned} \dot{\phi} &= -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 \\ \dot{\psi} &= \frac{1}{\beta^2}(\phi + 1 - u). \end{aligned}$$
Here ϕ and ψ are the deviations of the mass flow and the pressure rise from their set points, the control input u is the flow through the throttle, and β is a positive constant. Design a control law by backstepping to stabilize the origin $(\phi, \psi) = 0$ and show that the closed loop system is asymptotically stable with an appropriate Lyapunov function.

5. Determine whether the following scalar systems are input-to-state stable:

a) $\dot{x} = -(1+u)x^3$, b) $\dot{x} = -x + x^3 u$, c) $\dot{x} = -x^5 + u$, d) $\dot{x} = x - x^5 + u$.