Homework #9

Due: April 4, Wednesday

1. The so-called SIR model for epidemics is

$$\dot{S} = -\beta SI \quad \dot{I} = \beta SI - \gamma I \quad \dot{R} = \gamma I,$$

where S, I and R are the proportion of susceptible, infected, and recovered individuals in a population, $\beta > 0$ is the infection rate per contact, and $\gamma > 0$ is the recovery rate.

a) Show that $S(t) + I(t) + R(t)$ remains constant along the trajectories of the system. We take this constant to be 1, since S, I and R are proportions of the whole population.

b) Due to the constraint $S(t) + I(t) + R(t) = 1$ we can remove R from the variables of interest, and study the remaining equations for S and I as a planar system. Find all equilibrium points in the S-I plane and determine which ones are *unstable*.

c) Simulate the system with $\gamma = 1$, $I(0) = 0.001$, $S(0) = 1 - I(0)$, for $\beta = 1.3$, 1.6, 2, and 2.8. Superimpose the plots of $I(t)$ for each β, and comment on the effect of increasing β on the peak value of $I(t)$ and how soon the peak is reached.

2. Given the linear system $\dot{x} = Ax + Bu$ with initial condition $x(0) = 0$, and vectors a and b, we would like to check if the trajectories $x(t)$ generated by unit energy inputs ($\int_0^\infty u^T(t)u(t)dt \leq 1$) satisfy, for all $t \geq 0$,

$$|a^T x| \leq 1 \quad \text{and} \quad |b^T x| \leq 1. \tag{1}$$

a) Derive a linear matrix inequality (similar to those in Lecture 15) which, if feasible, certifies that the reachable set lies within the polyhedron defined by (1) above.

b) For the system below show that trajectories starting at $x(0) = 0$ subject to unit energy inputs satisfy $|x_1| \leq 1$ and $|x_2| \leq 1$ for all $t \geq 0$:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - 0.5x_2 + u.$$

3. (Sum-of-squares decomposition) In this problem you will show that

$$p(x, y) = x^4 + x^3y + x^2y^2 - xy^3 + y^4 \geq 0 \quad \text{for all } x, y.$$

a) Match the following expression to $p(x, y)$ by finding entries for $Q = Q^T$:

$$\begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}.$$

If the answer is not unique, then characterize all possible solutions.

b) Show that a *positive semidefinite* solution $Q = Q^T \geq 0$ exists within the set of matrices Q characterized in part (a), thus certifying $p(x, y) \geq 0$.

4. This exercise will familiarize you with the MATLAB package SOSOPT[1]. Complete the code in `exerciseSOS.m`[2] so that it runs to completion, and solves all posed problems.

5. (Region of attraction estimation) Consider the van der Pol oscillator in reverse time:

$$\dot{x}_1 = -x_2, \qquad \dot{x}_2 = x_1 - x_2 + x_2^3.$$

You will use SOSOPT to estimate the region of attraction (ROA) by finding a 4th-order polynomial Lyapunov function $V(x)$ and the largest level set of V in which $\nabla V(x)^T f(x)$ is negative definite. Complete the code in `vdpROAest.m`[3] which takes you through the following steps to obtain a tight estimate:

a) Let $V_0(x)$ be an initial choice for a Lyapunov function (*e.g.*, a quadratic function for the linearized model at the origin.) Find γ^* given by

$$\gamma^* := \max \gamma \quad \text{subject to} \quad \nabla V_0(x)^T f(x) < 0 \text{ whenever } x \neq 0 \text{ and } V_0(x) \leq \gamma.$$

This set containment problem can be relaxed with a generalization of the S-procedure:

$$\gamma^* := \max \gamma \quad \text{s.t.} \quad s_1(x) \quad \text{and} \quad -[\ell(x) + \nabla V_0(x)^T f(x)] - s_1(x)[\gamma^* - V_0(x)] \quad \text{are SOS}$$

where $l(x) := \epsilon(x_1^2 + x_2^2)$ with a small $\epsilon > 0$ and $s_1(x)$ is a SOS multiplier.

b) Let $p(x)$ be some fixed, positive definite convex polynomial (*e.g.*, $p(x) = x_1^2 + x_2^2$), and let $V_0(x)$ and γ^* be as in part (a). Find β^* given by

$$\beta^* := \max \beta \quad \text{subject to} \quad V_0(x) \leq \gamma^* \text{ whenever } p(x) \leq \beta.$$

This set containment problem is relaxed as follows, where $s_2(x)$ is a SOS multiplier:

$$\beta^* := \max \beta \quad \text{s.t.} \quad s_2(x) \quad \text{and} \quad [\gamma^* - V_0(x)] - s_2(x)[\beta - p(x)] \quad \text{are SOS.}$$

c) Given $\gamma^*, s_1(x)$ from part (a) and $p(x), s_2(x)$ from part (b), search for $V(x)$ to solve:

$$\max_{\beta > 0,\ \text{4th-order } V(x)} \beta \qquad \text{s.t.} \quad V(x) - \ell(x) \text{ is SOS}$$

$$-[\ell(x) + \nabla V(x)^T f(x)] - s_1(x)[\gamma^* - V(x)] \text{ is SOS}$$

$$[\gamma^* - V(x)] - s_2(x)[\beta - p(x)] \text{ is SOS.}$$

d) Replace $V_0(x)$ in part (a) with the function $V(x)$ from part (c), and repeat the steps above for several iterations, until the change in β^* in part (b) is sufficiently small. Plot the obtained approximation of the ROA (the set where $V(x) \leq \gamma^*$) along with the limit cycle of the van der Pol oscillator.

[1]Go to `http://www.aem.umn.edu/~AerospaceControl/` and download only the `SOSAnalysis` package under the MATLAB software section. This `SOSAnalysis` package includes `SOSOPT` as well as other supporting and demo files. Within the downloaded file, unzip the 5 compressed subfiles. Then run the `sosaddpath.m` file from MATLAB. The `SOSOPT` package uses the SeDuMi solver. Instructions for downloading and installing SeDuMi are mentioned on the website.

[2]The file `exerciseSOS.m` is available on bCourses under Files/HW.

[3]The template file `vdpROAest.m` is available on bCourses under Files/HW.