Linear Time-Varying Systems

\[ \dot{x} = A(t)x \quad x(t) = \Phi(t, t_0)x(t_0) \]  

(1)

- The state transition matrix \( \Phi(t, t_0) \) satisfies the equations:

\[ \frac{\partial}{\partial t} \Phi(t, t_0) = A(t)\Phi(t, t_0) \]  

(2)

\[ \frac{\partial}{\partial t_0} \Phi(t, t_0) = -\Phi(t, t_0)A(t_0) \]  

(3)

- No eigenvalue test for stability in the time-varying case:

\[ A(t) = \begin{bmatrix} -1 + 1.5 \cos^2 t & 1 - 1.5 \sin t \cos t \\ -1 - 1.5 \sin t \cos t & -1 + 1.5 \sin^2 t \end{bmatrix} \]

eigenvalues: \(-0.25 \pm i0.25\sqrt{7}\) for all \(t\), but unstable:

\[ \Phi(t, 0) = \begin{bmatrix} e^{0.5t} \cos t & e^{-t} \sin t \\ e^{-0.5t} \sin t & e^{-t} \cos t \end{bmatrix} \]

- For linear systems uniform asymptotic stability is equivalent to uniform exponential stability:

**Theorem**\(^2\): \(x = 0\) is uniformly asymptotically stable if and only if \[\|\Phi(t, t_0)\| \leq ke^{-\lambda(t-t_0)} \text{ for some } k > 0, \lambda > 0.\]

- Last lecture: \(V(t, x) = x^TP(t)x\) proves uniform exp. stability if

\[ (i) \quad \dot{P}(t) + A^T(t)P(t) + P(t)A(t) = -Q(t) \]

\[ (ii) \quad 0 < k_1 I \leq P(t) \leq k_2 I \]

\[ (iii) \quad 0 < k_3 I \leq Q(t) \text{ for all } t. \]

The converse is also true:

**Theorem**: Suppose \(x = 0\) is uniformly exponentially stable, \(A(t)\) is continuous and bounded, \(Q(t)\) is continuous and symmetric, and there exist \(k_3, k_4 > 0\) such that

\[ 0 < k_3 I \leq Q(t) \leq k_4 I \text{ for all } t. \]

Then, there exists a symmetric \(P(t)\) satisfying (i)–(ii) above.
Proof:

Time-invariant: \( P = \int_{0}^{\infty} e^{A\tau}Qe^{A\tau}d\tau \)

Time-varying: \( P(t) = \int_{t}^{\infty} \Phi^{T}(\tau,t)Q(\tau)\Phi(\tau,t)d\tau \)

Using the Leibniz rule, property (3), and \( \Phi(t,t) = I \) we obtain:

\[
\dot{P}(t) = \int_{t}^{\infty} \left( \frac{\partial}{\partial t} \Phi^{T}(\tau,t)Q(\tau)\Phi(\tau,t) + \Phi^{T}(\tau,t)Q(\tau)\frac{\partial}{\partial t} \Phi(\tau,t) \right) d\tau \\
= \int_{t}^{\infty} \left( -A^{T}(t)\Phi^{T}(\tau,t)Q(\tau)\Phi(\tau,t) - \Phi^{T}(\tau,t)Q(\tau)\Phi(\tau,t)A(t) \right) d\tau \\
= -A^{T}(t)P(t) - P(t)A(t) - Q(t).
\]

**Lyapunov-based Feedback Design Examples**

**Model Reference Adaptive Control**

Illustrated on a first order system:

\[
\dot{y} = a^{*}y + u
\]

where \( a^{*} \) is unknown.

Reference model:

\[
\dot{y}_{m} = -ay_{m} + r(t) \quad a > 0, \ r(t) : \text{reference signal.}
\]

**Goal:** Design a controller that guarantees \( y(t) - y_{m}(t) \to 0 \) without the knowledge of \( a^{*} \).

If we knew \( a^{*} \), we would choose:

\[
u = -(a^{*} + a)y + r(t) \quad \Rightarrow \quad \dot{y} = -ay + r(t).
\]

The tracking error \( e(t) := y(t) - y_{m}(t) \) then satisfies:

\[
\dot{e} = -ae \quad \Rightarrow \quad e(t) \to 0 \text{ exponentially.}
\]

**Adaptive design when** \( a^{*} \) (therefore, \( k^{*} \)) is unknown:

\[
u = -k(t)y + r(t)
\]

where \( \dot{k}(t) \) is to be designed. Then: \( \dot{e} = -ae - (k(t) - k^{*})y \).
Use the Lyapunov function: \( V = \frac{1}{2} e^2 + \frac{1}{2} \tilde{k}^2 \):

\[
V = -ae^2 - \tilde{k}ey + \tilde{k} \tilde{k} = -ae^2 + \tilde{k}(\tilde{k} - ey).
\]

Note \( \tilde{k} = k \) and choose \( \boxed{\tilde{k} = ey} \) so that \( V = -ae^2 \).

This guarantees stability of \((e, \tilde{k}) = (0, 0)\) and boundedness of \((e(t), \tilde{k}(t))\) since the level sets of \( V = \frac{1}{2} e^2 + \frac{1}{2} \tilde{k}^2 \) are positively invariant. In addition, if \( r(t) \) is bounded, then \( y_m(t) \) in (5) is bounded, and so is \( y(t) = y_m(t) + e(t) \). Then we can apply the Theorem from Lecture 11, page 3, to the time-varying model

\[
\dot{e} = -ae - y(t) \tilde{k}, \quad \dot{\tilde{k}} = y(t)e,
\]

and conclude from \( \dot{V} = -ae^2 \) that \( e(t) \to 0 \).

Whether \( \tilde{k}(t) \to 0 \) \( (k(t) \to k^*) \) depends on further properties of the reference signal \( r(\cdot) \) that are beyond the scope of this lecture.

**Backstepping**

Khalil (Sec. 14.3), Sastry (Sec. 6.8)

**Feedback stabilization:** Given the system

\[
\dot{x} = f(x) + g(x)u \tag{6}
\]

with input \( u \), design a control law \( u = \alpha(x) \) such that \( x = 0 \) is asymptotically stable for the closed-loop system:

\[
\dot{x} = f(x) + g(x)\alpha(x).
\]

Backstepping is a technique that simplifies this task for a class of systems.

Suppose a stabilizing feedback \( u = \alpha(X) \) is available for:

\[
\dot{X} = F(X) + G(X)u \quad X \in \mathbb{R}^n, u \in \mathbb{R}
\]

and suppose the closed-loop system admits a Lyapunov function \( V(X) \) such that

\[
\frac{\partial V}{\partial X} \left( F(X) + G(X)\alpha(X) \right) \leq -W(X) < 0 \quad \forall X \neq 0.
\]

Can we modify \( \alpha(X) \) to stabilize the augmented system below?

\[
\dot{X} = F(X) + G(X)x \\
\dot{x} = u.
\]

Define the error variable \( \boxed{z = x - \alpha(X)} \) and change variables:
\((X, x) \to (X, z)\):
\[
X = F(X) + G(X)\alpha(X) + G(X)z
\]
\[
\dot{z} = u - \dot{\alpha}(X, z)
\]

where \(\dot{\alpha}(X, z) = \frac{\partial}{\partial X} \left( F(X) + G(X)\alpha(X) + G(X)z \right)\). Take the new Lyapunov function:
\[
V_+(X, z) = V(X) + \frac{1}{2}z^2.
\]

\[
\dot{V}_+ = \frac{\partial V}{\partial X} \left( F(X) + G(X)\alpha(X) \right) + \frac{\partial V}{\partial X} G(X)z + z(u - \dot{\alpha})
\]
\[
\leq -W(X) = z\left( u - \dot{\alpha} + \frac{\partial V}{\partial X} G(X) \right)
\]

Let: \(u = \dot{\alpha} - \frac{\partial V}{\partial X} G(X) - kz, \quad k > 0\).

Then, \(\dot{V}_+ \leq -W(X) - kz^2 \Rightarrow (X, z) = 0\) is asymptotically stable.

**Example:**
\[
\begin{aligned}
\dot{x}_1 &= x_2^2 + x_2 \\
\dot{x}_2 &= u.
\end{aligned}
\]  \hspace{1cm} (7)

Treat \(x_2\) as “virtual” control input for the \(x_1\)-subsystem:
\[
\alpha(x_1) = -k_1x_1 - x_1^2 \quad k_1 > 0
\]
\[
V_1(x_1) = \frac{1}{2}x_1^2.
\]

Apply backstepping:
\[
\begin{aligned}
z_2 &= x_2 - \alpha(x_1) = x_2 + k_1x_1 + x_1^2 \\
\dot{z}_2 &= u - \dot{\alpha} \\
u &= \dot{\alpha} - \frac{\partial V_1}{\partial x_1} - k_2z_2, \quad k_2 > 0
\end{aligned}
\]
\[
= -(k_1 + 2x_1)(x_1^2 + x_2) - \frac{x_1}{\partial V_1/\partial x_1} - k_2(x_2 + k_1x_1 + x_1^2).
\]