# EE C222/ME C237 - Spring'18 - Lecture 12 Notes[1]

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## Linear Time-Varying Systems

Khalil Section 4.6, Sastry Section 5.7

$$\dot{x} = A(t)x \qquad x(t) = \Phi(t, t_0)x(t_0) \tag{1}$$

- The state transition matrix $\Phi(t, t_0)$ satisfies the equations:

$$\frac{\partial}{\partial t}\Phi(t, t_0) = A(t)\Phi(t, t_0) \tag{2}$$

$$\frac{\partial}{\partial t_0}\Phi(t, t_0) = -\Phi(t, t_0)A(t_0) \tag{3}$$

- No eigenvalue test for stability in the time-varying case:

$$A(t) = \begin{bmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t\cos t \\ -1 - 1.5\sin t\cos t & -1 + 1.5\sin^2 t \end{bmatrix}$$

  eigenvalues: $-0.25 \mp i0.25\sqrt{7}$ for all $t$, but unstable:

$$\Phi(t, 0) = \begin{bmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ e^{-0.5t}\sin t & e^{-t}\cos t \end{bmatrix}$$

- For linear systems uniform asymptotic stability is equivalent to uniform exponential stability:

  Theorem[2]: $x = 0$ is uniformly asymptotically stable if and only if

$$\|\Phi(t, t_0)\| \leq ke^{-\lambda(t-t_0)} \text{ for some } k > 0,\ \lambda > 0.$$

[2] Khalil Thm. 4.11, Sastry Thm. 5.33

- Last lecture: $V(t, x) = x^T P(t)x$ proves uniform exp. stability if

  (i) $\dot{P}(t) + A^T(t)P(t) + P(t)A(t) = -Q(t)$

  (ii) $0 < k_1 I \leq P(t) \leq k_2 I$

  (iii) $0 < k_3 I \leq Q(t)$ for all $t$.

  The converse is also true:

  Theorem: Suppose $x = 0$ is uniformly exponentially stable, $A(t)$ is continuous and bounded, $Q(t)$ is continuous and symmetric, and there exist $k_3, k_4 > 0$ such that

$$0 < k_3 I \leq Q(t) \leq k_4 I \text{ for all } t.$$

  Then, there exists a symmetric $P(t)$ satisfying (i)–(ii) above.

Proof:

Time-invariant: $P = \int_0^\infty e^{A^T\tau} Q e^{A\tau} d\tau$

Time-varying: $P(t) = \int_t^\infty \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)d\tau$

Using the Leibniz rule, property (3), and $\Phi(t,t) = I$ we obtain:

$$\begin{aligned}
\dot{P}(t) &= \int_t^\infty \left(\frac{\partial}{\partial t}\Phi^T(\tau,t)Q(\tau)\Phi(\tau,t) + \Phi^T(\tau,t)Q(\tau)\frac{\partial}{\partial t}\Phi(\tau,t)\right) d\tau \\
&\qquad - \Phi^T(t,t)Q(t)\Phi(t,t) \\
&= \int_t^\infty \left(-A^T(t)\Phi^T(\tau,t)Q(\tau)\Phi(\tau,t) - \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)A(t)\right) d\tau \\
&\qquad - \Phi^T(t,t)Q(t)\Phi(t,t) \\
&= -A^T(t)P(t) - P(t)A(t) - Q(t).
\end{aligned}$$

## *Lyapunov-based Feedback Design Examples*

### *Model Reference Adaptive Control*

Illustrated on a first order system:

$$\dot{y} = a^* y + u \tag{4}$$

where $a^*$ is unknown.

Reference model:

$$\dot{y}_m = -a y_m + r(t) \quad a > 0,\ r(t) : \text{reference signal.} \tag{5}$$

Goal: Design a controller that guarantees $y(t) - y_m(t) \to 0$ without the knowledge of $a^*$.

If we knew $a^*$, we would choose:

$$u = -\underbrace{(a^* + a)}_{=:k^*}y + r(t) \quad \Rightarrow \quad \dot{y} = -ay + r(t).$$

The tracking error $\boxed{e(t) := y(t) - y_m(t)}$ then satisfies:

$$\dot{e} = -ae \;\Rightarrow\; e(t) \to 0 \text{ exponentially.}$$

*Adaptive* design when $a^*$ (therefore, $k^*$) is unknown:

$$u = -k(t)y + r(t)$$

where $\dot{k}(t)$ is to be designed. Then: $\dot{e} = -ae - \underbrace{(k(t) - k^*)}_{=:\tilde{k}(t)}y.$

Use the Lyapunov function: $V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{k}^2$:

$$\begin{aligned}\dot{V} &= -ae^2 - \tilde{k}ey + \tilde{k}\dot{\tilde{k}} \\ &= -ae^2 + \tilde{k}(\dot{\tilde{k}} - ey).\end{aligned}$$

Note $\dot{\tilde{k}} = \dot{k}$ and choose $\boxed{\dot{k} = ey}$ so that $\dot{V} = -ae^2$.

This guarantees stability of $(e, \tilde{k}) = (0, 0)$ and boundedness of $(e(t), \tilde{k}(t))$ since the level sets of $V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{k}^2$ are positively invariant. In addition, if $r(t)$ is bounded, then $y_m(t)$ in (5) is bounded, and so is $y(t) = y_m(t) + e(t)$. Then we can apply the Theorem from Lecture 11, page 3, to the time-varying model

$$\dot{e} = -ae - y(t)\tilde{k}, \quad \dot{\tilde{k}} = y(t)e,$$

and conclude from $\dot{V} = -ae^2$ that $e(t) \to 0$.

Whether $\tilde{k}(t) \to 0$ ($k(t) \to k^*$) depends on further properties of the reference signal $r(\cdot)$ that are beyond the scope of this lecture.

*Backstepping*

Khalil (Sec. 14.3), Sastry (Sec. 6.8)

Feedback stabilization: Given the system

$$\dot{x} = f(x) + g(x)u \tag{6}$$

with input $u$, design a control law $u = \alpha(x)$ such that $x = 0$ is asymptotically stable for the closed-loop system:

$$\dot{x} = f(x) + g(x)\alpha(x).$$

Backstepping is a technique that simplifies this task for a class of systems.

Suppose a stabilizing feedback $u = \alpha(X)$ is available for:

$$\dot{X} = F(X) + G(X)u \quad X \in \mathbb{R}^n, u \in \mathbb{R}$$

and suppose the closed-loop system admits a Lyapunov function $V(X)$ such that

$$\frac{\partial V}{\partial X}\Big(F(X) + G(X)\alpha(X)\Big) \leq -W(X) < 0 \quad \forall X \neq 0.$$

Can we modify $\alpha(X)$ to stabilize the augmented system below?

$$\boxed{\begin{aligned}\dot{X} &= F(X) + G(X)x \\ \dot{x} &= u.\end{aligned}}$$

Define the error variable $\boxed{z = x - \alpha(X)}$ and change variables:

$(X, x) \to (X, z)$:

$$\dot{X} = F(X) + G(X)\alpha(X) + G(X)z$$
$$\dot{z} = u - \dot{\alpha}(X, z)$$

where $\dot{\alpha}(X,z) = \frac{\partial \alpha}{\partial X}\Big(F(X) + G(X)\alpha(X) + G(X)z\Big)$. Take the new Lyapunov function:

$$V_+(X, z) = V(X) + \frac{1}{2}z^2.$$

$$\dot{V}_+ = \underbrace{\frac{\partial V}{\partial X}\Big(F(X) + G(X)\alpha(X)\Big)}_{\leq -W(X)} + \underbrace{\frac{\partial V}{\partial X}G(X)z + z(u - \dot{\alpha})}_{= z\left(u - \dot{\alpha} + \frac{\partial V}{\partial X}G(X)\right)}$$

Let: $\boxed{u = \dot{\alpha} - \frac{\partial V}{\partial X}G(X) - kz, \quad k > 0.}$

Then, $\dot{V}_+ \leq -W(X) - kz^2 \;\Rightarrow\; (X, z) = 0$ is asymptotically stable.

Example:

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2 \\ \dot{x}_2 &= u. \end{aligned} \tag{7}$$

Treat $x_2$ as "virtual" control input for the $x_1$-subsystem:

$$\alpha(x_1) = -k_1 x_1 - x_1^2 \quad k_1 > 0$$
$$V_1(x_1) = \frac{1}{2}x_1^2.$$

Apply backstepping:

$$z_2 = x_2 - \alpha(x_1) = x_2 + k_1 x_1 + x_1^2$$
$$\dot{z}_2 = u - \dot{\alpha}$$
$$u = \dot{\alpha} - \frac{\partial V_1}{\partial x_1} - k_2 z_2, \quad k_2 > 0$$
$$= \underbrace{-(k_1 + 2x_1)(x_1^2 + x_2)}_{=\dot{\alpha}} - \underbrace{x_1}_{=\frac{\partial V_1}{\partial x_1}} - k_2\underbrace{(x_2 + k_1 x_1 + x_1^2)}_{= z_2}.$$