Backstepping

Suppose a stabilizing feedback \( u = \alpha(X), \alpha(0) = 0 \), is available for:
\[
\dot{X} = F(X) + G(X)u \quad X \in \mathbb{R}^n, u \in \mathbb{R}, \quad F(0) = 0,
\]
along with a Lyapunov function \( V \) such that
\[
\frac{\partial V}{\partial X} \left( F(X) + G(X)\alpha(X) \right) \leq -W(X) < 0 \quad \forall X \neq 0.
\]
Can we modify \( \alpha(X) \) to stabilize the augmented system below?
\[
\dot{X} = F(X) + G(X)\alpha(X)
\]
\[
\dot{x} = u.
\]

Define the error variable \( z = x - \alpha(X) \) and change variables:

\[
(X, x) \rightarrow (X, z):
\]
\[
\dot{X} = F(X) + G(X)\alpha(X) + G(X)z
\]
\[
\dot{z} = u - \dot{\alpha}(X, z)
\]

where \( \dot{\alpha}(X, z) = \frac{\partial \alpha}{\partial X} \left( F(X) + G(X)\alpha(X) + G(X)z \right) \). Take the new Lyapunov function:
\[
V_+(X, z) = V(X) + \frac{1}{2}z^2.
\]
\[
\dot{V}_+ = \frac{\partial V}{\partial X} \left( F(X) + G(X)\alpha(X) \right) + \frac{\partial V}{\partial X} G(X)z + z(u - \dot{\alpha})
\]
\[
\leq -W(X) + z \left( u - \dot{\alpha} + \frac{\partial V}{\partial X} G(X) \right)
\]

Let:
\[
u = \dot{\alpha} - \frac{\partial V}{\partial X} G(X) - k z, \quad k > 0.
\]

Then, \( \dot{V}_+ \leq -W(X) - kz^2 \Rightarrow (X, z) = 0 \) is asymptotically stable.

Example 1:
\[
\begin{align*}
\dot{x}_1 &= x_1^2 + x_2 \\
\dot{x}_2 &= u.
\end{align*}
\]

Treat \( x_2 \) as “virtual” control input for the \( x_1 \)-subsystem:
\[
\alpha(x_1) = -k_1 x_1 - x_1^2 \quad k_1 > 0
\]
\[
V_1(x_1) = \frac{1}{2} x_1^2.
\]
Apply backstepping:

\[ z_2 = x_2 - a(x_1) = x_2 + k_1 x_1 + x_1^2 \]
\[ \dot{z}_2 = u - \dot{a} \]
\[ u = \dot{a} - \frac{\partial V_1}{\partial x_1} - k_2 z_2, \quad k_2 > 0 \]
\[ = -(k_1 + 2x_1)(x_1^2 + x_2) - \frac{x_1}{\frac{\partial V_1}{\partial x_1}} - k_2(x_2 + k_1 x_1 + x_1^2). \]

- Above we discussed backstepping over a pure integrator. The main idea generalizes trivially to:

\[
\begin{align*}
\dot{X} &= F(X) + G(X)x \\
\dot{x} &= f(X, x) + g(X, x)u
\end{align*}
\]

where \( X \in \mathbb{R}^n \), \( x \in \mathbb{R} \), and \( g(X, x) \neq 0 \) for all \((X, x) \in \mathbb{R}^{n+1}\).

With the preliminary feedback

\[ u = \frac{1}{g(X, x)} (-f(X, x) + v) \] (2)

the \( x \)-subsystem becomes a pure integrator: \( \dot{x} = v \). Substituting the backstepping control law from above:

\[ v = \dot{a} - \frac{\partial V}{\partial X} G(X) - kz, \quad z \triangleq x - a(X), \quad k > 0 \]

into (2), we get:

\[ u = \frac{1}{g(X, x)} \left( -f(X, x) + \dot{a} - \frac{\partial V}{\partial X} G(X) - kz \right). \]

- Backstepping can be applied recursively to systems of the form:\(^2\)

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\
\dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)x_4 \\
&\quad \vdots \\
\dot{x}_n &= f_n(x) + g_n(x)u
\end{align*}
\] (3)

where \( g_i(x_1, \ldots, x_i) \neq 0 \) for all \( x \in \mathbb{R}^n, i = 1, 2, \ldots, n \).

Example 2:  \[ \begin{align*}
\dot{x}_1 &= (x_1 x_2 - 1)x_1^3 + (x_1 x_2 + x_3^2 - 1)x_1 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u
\end{align*} \] (4)

\(^2\) Systems of this form are called “strict feedback systems.”
Not in strict feedback form because $x_3$ appears too soon. In fact, this system is not globally stabilizable because the set $x_1 x_2 \geq 2$ is positively invariant regardless of $u$:

$$n(x) = \begin{bmatrix} x_2 \\ x_1 \\ 0 \end{bmatrix}$$

To see this, note that

$$n(x) \cdot f(x, u) = [(x_1 x_2 - 1)x_1^3 + (x_1 x_2 + x_2^2 - 1)x_1]x_2 + x_3 x_1$$

and substitute $x_1 x_2 = 2$:

$$= \left( x_1^3 + (1 + x_3^3) x_1 \right) x_2 + x_3 x_1$$
$$= \left( x_1^3 + (1 + x_3^3) \right) x_1 x_2 + x_3 x_1$$
$$= 2x_1^3 + 2(1 + x_3^3) + x_3 x_1$$
$$= 2x_1^3 + x_3 x_1 + 2x_3^2 + 2 > 0.$$  

The condition $g_i(x_1, \ldots, x_i) \neq 0$ in (3) can be relaxed in some cases:

**Example 3:**

$$\dot{x}_1 = x_1^2 x_2$$
$$\dot{x}_2 = u \quad (5)$$

Treat $x_2$ as virtual control and let $\alpha_1(x_1) = -x_1$ which stabilizes the $x_1$-subsystem, as verified with Lyapunov function $V_1(x_1) = \frac{1}{2} x_1^2$.

Then $z_2 := x_2 - \alpha_1(x_1)$ satisfies $\dot{z}_2 = u - \dot{\alpha}_1$, and

$$u = \dot{\alpha}_1 - \frac{\partial V_1}{\partial x_1} x_1^2 - k_2 z_2 = -x_1^2 x_2 - x_3^2 - k_2 (x_2 + x_1)$$

achieves global asymptotic stability:

$$V = \frac{1}{2} x_1^2 + \frac{1}{2} z_2^2 \implies \dot{V} = -x_1^4 - k_2 z_2^2.$$  

Note that we can’t conclude exponential stability due to the quartic term $x_1^4$ above (recall the Lyapunov sufficient condition for exponential stability in Lecture 11, p.2). In fact, the linearization of the
closed-loop system proves the lack of exponential stability:

\[
\begin{bmatrix}
0 & 0 \\
0 & -k_2
\end{bmatrix} \rightarrow \lambda_{1,2} = 0, -k_2.
\]

**Design example:** Active suspension

![Active Suspension Diagram](image)

Design example: Active suspension

Krstić et al., Nonlinear and Adaptive Control Design, Section 2.2.2.

Define state variables: \( x_1 = x_s, x_2 = \dot{x}_s, x_3 = x_a, x_4 = Q \):

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k_a}{M_b} (x_1 - x_3) - \frac{c_a}{M_b} (x_2 - \frac{1}{A} x_4) \\
\dot{x}_3 &= \frac{1}{A} x_4 \\
\dot{x}_4 &= -c_f x_4 + k_f u.
\end{align*}
\] (6)

This system is not in strict feedback form due to the \( x_4 \) term in \( \dot{x}_2 \). To overcome this problem define:

\[
\begin{align*}
\tilde{x}_3 &= k_a M_b x_3 + c_a M_b x_4 \\
\xi &= x_3
\end{align*}
\]

and change variables to \((x_1, x_2, \tilde{x}_3, \xi)\):

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k_a}{M_b} x_1 - \frac{c_a}{M_b} x_2 + \tilde{x}_3 \\
\dot{\tilde{x}}_3 &= \frac{k_a - c_a c_f}{M_b A} x_4 + \frac{c_a k_f}{M_b A} u.
\end{align*}
\]
Two steps of backstepping starting with the virtual control law:

\[ a_1(x_1) = -c_1 x_1 - k_1 x_1^3 \]

will stabilize the \((x_1, x_2, \bar{x}_3)\) subsystem. Full \((x_1, x_2, \bar{x}_3, \xi)\) system:

\[ \bar{x}_3 \]

The stiff nonlinearity \(k_1 x_1^3\) prevents large excursions of \(x_1\).

\[ \dot{\xi} = -\frac{k_x}{M_p} \bar{x}_3 + \frac{1}{\lambda} \bar{x}_3 \]

The \(\xi\)-subsystem is an asymptotically stable linear system driven by \(\bar{x}_3\); therefore the full system is stabilized.