# *EE C222/ME C237 - Spring'18 - Lecture 13 Notes*[1]

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## *Backstepping*

Khalil (Sec. 14.3), Sastry (Sec. 6.8)

Suppose a stabilizing feedback $u = \alpha(X)$, $\alpha(0) = 0$, is available for:

$$\dot{X} = F(X) + G(X)u \quad X \in \mathbb{R}^n, u \in \mathbb{R}, \; F(0) = 0,$$

along with a Lyapunov function $V$ such that

$$\frac{\partial V}{\partial X}\Big(F(X) + G(X)\alpha(X)\Big) \leq -W(X) < 0 \quad \forall X \neq 0.$$

Can we modify $\alpha(X)$ to stabilize the augmented system below?

$$\boxed{\begin{aligned} \dot{X} &= F(X) + G(X)x \\ \dot{x} &= u. \end{aligned}}$$

Define the error variable $\boxed{z = x - \alpha(X)}$ and change variables: $(X, x) \to (X, z)$:

$$\begin{aligned} \dot{X} &= F(X) + G(X)\alpha(X) + G(X)z \\ \dot{z} &= u - \dot{\alpha}(X, z) \end{aligned}$$

where $\dot{\alpha}(X, z) = \frac{\partial \alpha}{\partial X}\Big(F(X) + G(X)\alpha(X) + G(X)z\Big)$. Take the new Lyapunov function:

$$V_+(X, z) = V(X) + \frac{1}{2}z^2.$$

$$\dot{V}_+ = \underbrace{\frac{\partial V}{\partial X}\Big(F(X) + G(X)\alpha(X)\Big)}_{\leq -W(X)} + \underbrace{\frac{\partial V}{\partial X}G(X)z + z(u - \dot{\alpha})}_{= z\left(u - \dot{\alpha} + \frac{\partial V}{\partial X}G(X)\right)}$$

Let: $\boxed{u = \dot{\alpha} - \frac{\partial V}{\partial X}G(X) - kz, \quad k > 0.}$

Then, $\dot{V}_+ \leq -W(X) - kz^2 \;\Rightarrow\; (X, z) = 0$ is asymptotically stable.

Example 1:

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2 \\ \dot{x}_2 &= u. \end{aligned} \tag{1}$$

Treat $x_2$ as "virtual" control input for the $x_1$-subsystem:

$$\begin{aligned} \alpha(x_1) &= -k_1 x_1 - x_1^2 \quad k_1 > 0 \\ V_1(x_1) &= \frac{1}{2}x_1^2. \end{aligned}$$

Apply backstepping:

$$\begin{aligned} z_2 &= x_2 - \alpha(x_1) = x_2 + k_1 x_1 + x_1^2 \\ \dot{z}_2 &= u - \dot{\alpha} \\ u &= \dot{\alpha} - \frac{\partial V_1}{\partial x_1} - k_2 z_2, \quad k_2 > 0 \\ &= \underbrace{-(k_1 + 2x_1)(x_1^2 + x_2)}_{=\dot{\alpha}} - \underbrace{x_1}_{=\frac{\partial V_1}{\partial x_1}} - k_2 \underbrace{(x_2 + k_1 x_1 + x_1^2)}_{=z_2}. \end{aligned}$$

- Above we discussed backstepping over a pure integrator. The main idea generalizes trivially to:

$$\boxed{\begin{aligned} \dot{X} &= F(X) + G(X)x \\ \dot{x} &= f(X, x) + g(X, x)u \end{aligned}}$$

where $X \in \mathbb{R}^n$, $x \in \mathbb{R}$, and $g(X, x) \neq 0$ for all $(X, x) \in \mathbb{R}^{n+1}$.

With the preliminary feedback

$$u = \frac{1}{g(X, x)}(-f(X, x) + v) \tag{2}$$

the $x$-subsystem becomes a pure integrator: $\dot{x} = v$. Substituting the backstepping control law from above:

$$v = \dot{\alpha} - \frac{\partial V}{\partial X} G(X) - kz, \quad z \triangleq x - \alpha(X), \quad k > 0$$

into (2), we get:

$$\boxed{u = \frac{1}{g(X, x)} \left( -f(X, x) + \dot{\alpha} - \frac{\partial V}{\partial X} G(X) - kz \right).}$$

- Backstepping can be applied recursively to systems of the form:[2]

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\ \dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)x_4 \\ &\vdots \\ \dot{x}_n &= f_n(x) + g_n(x)u \end{aligned} \tag{3}$$

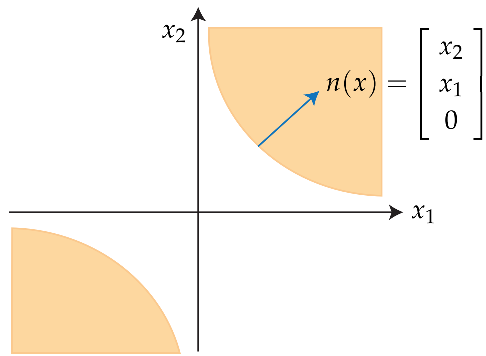
where $g_i(x_1, \ldots, x_i) \neq 0$ for all $x \in \mathbb{R}^n$, $i = 1, 2, \cdots, n$.

[2] Systems of this form are called "strict feedback systems."

Example 2:

$$\begin{aligned} \dot{x}_1 &= (x_1 x_2 - 1)x_1^3 + (x_1 x_2 + x_3^2 - 1)x_1 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u. \end{aligned} \tag{4}$$

Not in strict feedback form because $x_3$ appears too soon. In fact, this system is not globally stabilizable because the set $x_1 x_2 \geq 2$ is positively invariant regardless of $u$:




To see this, note that

$$n(x) \cdot f(x,u) = [(x_1x_2 - 1)x_1^3 + (x_1x_2 + x_3^2 - 1)x_1]x_2 + x_3x_1$$

and substitute $x_1 x_2 = 2$ :

$$\begin{aligned} &= \left(x_1^3 + (1 + x_3^2)x_1\right)x_2 + x_3x_1 \\ &= \left(x_1^2 + (1 + x_3^2)\right)x_1x_2 + x_3x_1 \\ &= 2x_1^2 + 2(1 + x_3^2) + x_3x_1 \\ &= \underbrace{2x_1^2 + x_3x_1 + 2x_3^2}_{\geq 0} + 2 > 0. \end{aligned}$$

- The condition $g_i(x_1, \ldots, x_i) \neq 0$ in (3) can be relaxed in some cases:

Example 3:

$$\begin{aligned} \dot{x}_1 &= x_1^2 x_2 \\ \dot{x}_2 &= u \end{aligned} \tag{5}$$

Treat $x_2$ as virtual control and let $\boxed{\alpha_1(x_1) = -x_1}$ which stabilizes the $x_1$-subsystem, as verified with Lyapunov function $V_1(x_1) = \frac{1}{2}x_1^2$. Then $z_2 := x_2 - \alpha_1(x_1)$ satisfies $\dot{z}_2 = u - \dot{\alpha}_1$, and

$$u = \dot{\alpha}_1 - \frac{\partial V_1}{\partial x_1} x_1^2 - k_2 z_2 = -x_1^2 x_2 - x_1^3 - k_2(x_2 + x_1)$$

achieves global asymptotic stability:

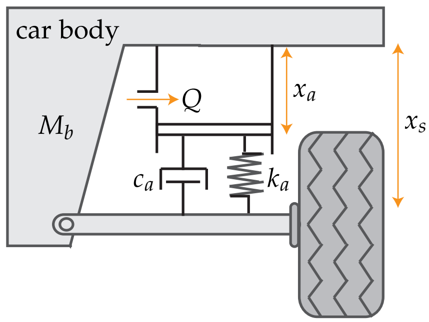
$$V = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2 \quad \Rightarrow \quad \dot{V} = -x_1{}^4 - k_2 z_2^2.$$

Note that we can't conclude exponential stability due to the quartic term $x_1^4$ above (recall the Lyapunov sufficient condition for exponential stability in Lecture 11, p.2). In fact, the linearization of the closed-loop system proves the lack of exponential stability:

$$\begin{bmatrix} 0 & 0 \\ 0 & -k_2 \end{bmatrix} \rightarrow \lambda_{1,2} = 0, -k_2.$$

Design example: Active suspension

Krstić et al., Nonlinear and Adaptive Control Design, Section 2.2.2.




$$M_b \ddot{x}_s = -k_a(x_s - x_a) - c_a(\dot{x}_s - \dot{x}_a)$$

$$\dot{x}_a = \frac{1}{A} Q \qquad A\text{: effective piston surface}$$

$$\text{Flow: } \dot{Q} = -c_f Q + k_f u \qquad u\text{: current applied to the solenoid valve (control input)}$$

Define state variables: $x_1 = x_s$, $x_2 = \dot{x}_s$, $x_3 = x_a$, $x_4 = Q$:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k_a}{M_b}(x_1 - x_3) - \frac{c_a}{M_b}(x_2 - \frac{1}{A} x_4) \\
\dot{x}_3 &= \frac{1}{A} x_4 \\
\dot{x}_4 &= -c_f x_4 + k_f u.
\end{aligned} \tag{6}$$

This system is not in strict feedback form due to the $x_4$ term in $\dot{x}_2$. To overcome this problem define:

$$\begin{aligned}
\bar{x}_3 &\triangleq \frac{k_a}{M_b} x_3 + \frac{c_a}{M_b A} x_4 \\
\xi &\triangleq x_3
\end{aligned}$$

and change variables to $(x_1, x_2, \bar{x}_3, \xi)$:

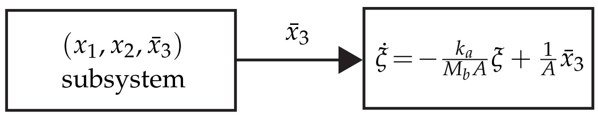
$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k_a}{M_b} x_1 - \frac{c_a}{M_b} x_2 + \bar{x}_3 \\
\dot{\bar{x}}_3 &= \frac{k_a - c_a c_f}{M_b A} x_4 + \frac{c_a k_f}{M_b A} u.
\end{aligned}$$

Two steps of backstepping starting with the virtual control law:

$$\alpha_1(x_1) = -c_1 x_1 - k_1 x_1^3$$

The stiff nonlinearity $k_1 x_1^3$ prevents large excursions of $x_1$.

will stabilize the $(x_1, x_2, \bar{x}_3)$ subsystem. Full $(x_1, x_2, \bar{x}_3, \xi)$ system:




The $\xi$-subsystem is an asymptotically stable linear system driven by $\bar{x}_3$; therefore the full system is stabilized.