

EE C222/ME C237 - Spring'18 - Lecture 14 Notes¹

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Input-to-State Stability

$$\dot{x} = f(x, u) \quad u: \text{exogenous input}$$

For linear systems, asymp. stability of the zero-input model $\dot{x} = Ax$ implies a bounded-input bounded-state property for $\dot{x} = Ax + Bu$:

$$\begin{aligned} x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ \implies |x(t)| &\leq \|e^{At}\| |x_0| + \int_0^t \|e^{A(t-\tau)}\| \|B\| |u(\tau)| d\tau \\ &\leq \kappa e^{-\alpha t} |x_0| + \|B\| \sup_{0 \leq \tau \leq t} |u(\tau)| \int_0^t \kappa e^{-\alpha(t-\tau)} d\tau \\ &\leq \underbrace{\kappa e^{-\alpha t} |x_0|}_{\text{effect of initial condition}} + \underbrace{\frac{\kappa}{\alpha} \|B\| \sup_{0 \leq \tau \leq t} |u(\tau)|}_{\text{effect of input}}. \end{aligned}$$

Khalil, Section 4.9

For nonlinear systems $\dot{x} = f(x, u)$, asymp. stability of the origin for the zero-input model $\dot{x} = f(x, 0)$ does not guarantee boundedness of states under bounded inputs.

Example 1: $\dot{x} = -x + xu$

$u(t) \equiv \text{constant} > 1 \implies$ exponential growth of $x(t)$.

A precise formulation of the bounded-input bounded-state property for nonlinear systems:

Definition: The system $\dot{x} = f(x, u)$, $f(0, 0) = 0$ is said to be **input-to-state stable (ISS)** if:

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma \left(\sup_{0 \leq \tau \leq t} |u(\tau)| \right)$$

for some class- \mathcal{KL} function β and class- \mathcal{K} function γ , called an ISS gain function.

Example: For the linear system above, $\gamma(s) = \frac{\kappa}{\alpha} \|B\| s$.

Implications of ISS

1. $\dot{x} = f(x, u)$ ISS $\implies \dot{x} = f(x, 0)$ globally asymptotically stable

Proof:

Substitute $u(t) \equiv 0$ in the definition above: $|x(t)| \leq \beta(|x(0)|, t)$.

2. $u(t) \rightarrow 0$ as $t \rightarrow \infty \Rightarrow x(t) \rightarrow 0$ as $t \rightarrow \infty$.

Proof:

Need to show that for any $\epsilon > 0$, there exists T such that

$$|x(t)| \leq \epsilon \quad \forall t \geq T.$$

Since $u(t) \rightarrow 0$, we can find T_1 such that $\gamma(|u(t)|) \leq \epsilon/2$ for all $t \geq T_1$. Choose $t_0 = T_1$ and apply ISS definition:

$$|x(t)| \leq \beta(|x(T_1)|, t - T_1) + \epsilon/2 \quad \forall t \geq T_1.$$

Choose T_2 such that

$$\beta(|x(T_1)|, T_2) \leq \epsilon/2.$$

Then, $|x(t)| \leq \epsilon$ for all $t \geq T_1 + T_2 \triangleq T$.

A Lyapunov Characterization of ISS

The system $\dot{x} = f(x, u)$ is ISS if there exist class- \mathcal{K}_∞ functions α_i , $i = 1, 2, 3, 4$, and a C^1 function V such that

$$\begin{aligned} \alpha_1(|x|) &\leq V(x) \leq \alpha_2(|x|) \\ \frac{\partial V}{\partial x} f(x, u) &\leq -\alpha_3(|x|) + \alpha_4(|u|). \end{aligned}$$

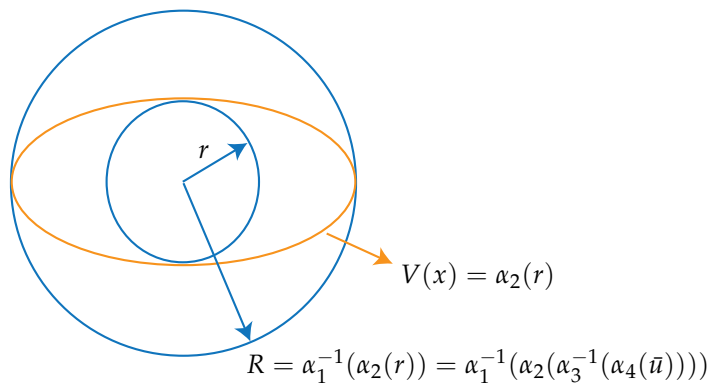
V is called an "ISS Lyapunov function."

Sketch of the proof:

Let $\bar{u} \triangleq \sup_{\tau \geq 0} |u(\tau)|$. Then:

$$|x| \geq r \triangleq \alpha_3^{-1}(\alpha_4(\bar{u})) \Rightarrow \frac{\partial V}{\partial x} f(x, u(t)) \leq 0 \quad \forall t \geq 0.$$

This implies that the level set $\{x : V(x) \leq \alpha_2(r)\}$ is invariant and attractive. Thus, all trajectories converge to this level set which is enclosed in the outer ball $|x| \leq R \triangleq \alpha_1^{-1}(\alpha_2(r))$.



Example 2: $\dot{x} = -x^r + x^s u$, r : odd integer, is ISS if $r > s$. Take:

$$V(x) = \frac{1}{2}x^2$$

$$\dot{V}(x) = -x^{r+1} + x^{s+1}u.$$

Young's inequality (recall from homework):

$$yz \leq \frac{\lambda^p}{p}|y|^p + \frac{1}{q\lambda^q}|z|^q$$

for any $\lambda > 0$, and $p > 1, q > 1$ satisfying $(p-1)(q-1) = 1$. Apply to:

$$x^{s+1}u \leq \frac{\lambda^p}{p}|x|^{(s+1)p} + \frac{1}{q\lambda^q}|u|^q$$

and choose

$$p = \frac{r+1}{s+1} \quad q = 1 + \frac{1}{p-1} \quad \text{and } \lambda \text{ such that } \frac{\lambda^p}{p} = \frac{1}{2}$$

$$\Rightarrow x^{s+1}u \leq \frac{1}{2}|x|^{r+1} + \frac{1}{q\lambda^q}|u|^q$$

$$\Rightarrow \dot{V}(x) \leq -|x|^{r+1} + \frac{1}{2}|x|^{r+1} + \frac{1}{q\lambda^q}|u|^q$$

$$\leq \underbrace{-\frac{1}{2}|x|^{r+1}}_{-\alpha_3(|x|)} + \underbrace{\frac{1}{q\lambda^q}|u|^q}_{-\alpha_4(|u|)}.$$

Note:

- $\dot{x} = -x + xu$ ($r = s = 1$) is not ISS as shown in Example 1.
- $\dot{x} = -x + x^2u$ ($r = 1, s = 2$) is not ISS: it exhibits finite time escape for $u(t) \equiv \text{constant} \neq 0$, even with an exponentially decaying $u(t)$.
- $\dot{x} = -x^3 + u$ ($r = 3, s = 0$) is ISS.

Example 3:

$$\dot{x}_1 = -x_1 + x_2^2$$

$$\dot{x}_2 = -x_2 + u.$$

Let $V(x) = \frac{1}{2}x_1^2 + \frac{a}{4}x_2^4$, $a > 0$ to be determined.²

² Why not quadratic V?

$$\dot{V}(x) = -x_1^2 + x_1x_2^2 + a(-x_2^4 + x_2^3u)$$

Apply the Young Inequalities:

$$x_1x_2^2 \leq \frac{1}{2}x_1^2 + \frac{1}{2}x_2^4$$

$$x_2^3u \leq \frac{\lambda^{4/3}}{4/3}x_2^4 + \frac{1}{4\lambda^4}u^4$$

Choose λ such that $\frac{\lambda^{4/3}}{4/3} = \frac{1}{2}$.

$$\dot{V}(x) \leq -\frac{1}{2}x_1^2 + \frac{1}{2}x_2^4 + a \left(-\frac{1}{2}x_2^4 + \frac{1}{4\lambda^4}u^4 \right)$$

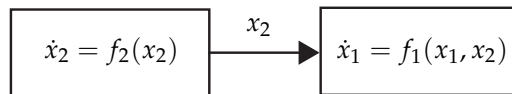
Let $a = 2$:

$$\dot{V}(x) \leq \underbrace{-\frac{1}{2}x_1^2 - \frac{1}{2}x_2^4}_{\leq -\alpha_3(|x|)} + \underbrace{\frac{1}{2\lambda^4}u^4}_{=\alpha_4(|u|)}$$

for an appropriate choice of α_3 . Thus, the system is ISS.

Stability of Series Interconnections

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) & x_1 &\in \mathbb{R}^{n_1} \\ \dot{x}_2 &= f_2(x_2) & x_2 &\in \mathbb{R}^{n_2} \end{aligned} \quad (1)$$



Suppose $x_2 = 0$ is globally asymptotically stable for $\dot{x}_2 = f_2(x_2)$ and $x_1 = 0$ is globally asymptotically stable for $\dot{x}_1 = f_1(x_1, 0)$. Is $(x_1, x_2) = 0$ globally asymptotically stable for the interconnection?

Answer: No.

Example 4:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 x_2 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

exhibits finite time escape.

Proposition: Consider the series interconnection:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_2, u). \end{aligned}$$

If the x_1 subsystem is ISS with x_2 viewed as an input, and the x_2 subsystem is ISS with input u , then the interconnection is ISS.

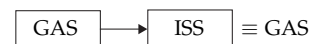
Example 3 revisited:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 & \text{is ISS with respect to } x_2 \\ \dot{x}_2 &= -x_2 + u & \text{is ISS with input } u \end{aligned}$$

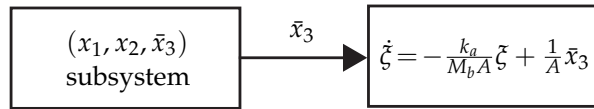
\Rightarrow the interconnection is ISS — an alternative to the proof in Ex. 3.

Corollary: $(x_1, x_2) = 0$ is globally asymptotically stable when $u \equiv 0$.

Note that Example 4 fails the ISS condition for the x_1 subsystem.



Example: Active suspension design example in Lecture 13:



The (x_1, x_2, \bar{x}_3) -subsystem globally asymptotically stabilized by back-stepping. The ζ -subsystem is an asymptotically stable linear system, therefore ISS with respect to the input \bar{x}_3 .