EE C222/ME C237 - Spring'18 - Lecture 16 Notes[1]

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Sum of Squares Programming

Establishing nonnegativity of functions is critical in nonlinear system analysis, *e.g.*, a Lyapunov function V for $\dot{x} = f(x)$ must satisfy

$$V(x) > 0 \quad \forall x \neq 0 \tag{1}$$

$$-\nabla V(x)^T f(x) \geq 0 \quad \forall x. \tag{2}$$

For $f(x) = Ax$ and $V(x) = x^T P x$, the conditions above are simple matrix inequalities:

$$P > 0, \quad -A^T P - PA \geq 0.$$

How can we check nonnegativity when f and V are more general polynomials?

Sum of Squares (SOS) Polynomials

A *monomial* is a product of powers of variables (*e.g.*, $m(x) = x_1^2 x_2$) and its degree is the sum of its exponents (*e.g.*, 3 for $m(x) = x_1^2 x_2$).

A *polynomial* is a finite linear combination of monomials and its degree is the maximum degree of these monomials.

Example 1: The polynomial

$$q(x_1, x_2) = x_1^2 - 2x_1 x_2^2 + 2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 6x_2^4 \tag{3}$$

has degree 4.

Definition: A polynomial p is a *sum of squares* (SOS) if there exist polynomials $g_1, \cdots, g_r$ such that

$$p = \sum_{i=1}^{r} g_i^2. \tag{4}$$

A SOS polynomial $p(x)$ is nonnegative for all x. The converse is not true: there exist nonnegative polynomials that are not SOS.

The polynomial $q(x_1, x_2)$ in (3) is SOS because it can be rewritten as:

$$(x_1 - x_2^2)^2 + \frac{1}{2}\left(2x_1^2 + x_1 x_2 - 3x_2^2\right)^2 + \frac{1}{2}\left(3x_1 x_2 + x_2^2\right)^2. \tag{5}$$

You can verify the equivalence of (3) and (5) by multiplying out terms in (5) and matching them to those in (3).

How a SOS decomposition like (5) can be obtained is discussed next.

SOS Decomposition

Let $z(x)$ be the vector of all monomials of degree $\leq d$ in n variables[2]:

$$z(x) \triangleq [1, x_1, x_2, \dots, x_n, x_1^2, x_1x_2, \dots, x_n^d]^T.$$

Then any polynomial with degree $\leq 2d$ can be rewritten as

$$p(x) = z(x)^T Q\, z(x) \tag{6}$$

where Q is a symmetric matrix.

Example 2: Let $p(x_1, x_2) = 2x_1^2x_2^2$ which has degree 4. With $n = 2$ and $d = 2$,

$$z(x) = [1, x_1, x_2, x_1^2, x_1x_2, x_2^2]^T, \tag{7}$$

and (6) holds with either

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Thus, the choice of Q is not unique.

Theorem: A polynomial p with degree $\leq 2d$ is SOS if and only if there exists $Q = Q^T \geq 0$ satisfying (6).

Proof: (only if) If p is SOS then, by definition, $p = \sum_{i=1}^r g_i^2$ for some polynomials g_i, $i = 1, \cdots, r$. Write g_i as:

$$g_i(x) = C_i z(x) \tag{8}$$

where C_i is a row vector of coefficients. Then $g_i^2 = z^T C_i^T C_i z$ and

$$p = \sum_{i=1}^r g_i^2 = z^T \underbrace{\left(\sum_{i=1}^r C_i^T C_i \right)}_{Q \geq 0} z.$$

(if) Given $Q = Q^T \geq 0$ satisfying (6), decompose Q as $Q = C^T C$ where C has as many rows as the rank of Q, say r. Then,

$$Q = C^T C = \sum_{i=1}^r C_i^T C_i$$

where C_i is the ith row. If we define g_i as in (8), then $z^T Q z = \sum_{i=1}^r g_i^2$. □

Since Q is not unique, not all Q satisfying (6) will certify SOS. In Example 2 above, $Q_1 \geq 0$ but Q_2 is indefinite. We need to characterize the set of all Q satisfying (6) and search for a $Q \geq 0$ in this set.

[2] The length of this vector is $l_{[n,d]} := \binom{n+d}{d}$.

Parameterization of all matrices Q satisfying (6):

Find a particular solution Q_0 such that

$$p(x) = z(x)^T Q_0 z(x),$$

and find a basis of symmetric matrices N_j, $j = 1, 2, \cdots, K$, such that[3]

$$z(x)^T N_j z(x) = 0 \quad \text{for all } x. \tag{9}$$

[3] There are $K = \frac{l_{[n,d]}(l_{[n,d]}+1)}{2} - l_{[n,2d]}$ such matrices.

Then we can parameterize the set of all Q satisfying (6) as

$$Q = Q_0 + \sum_{j=1}^{K} \lambda_j N_j \quad \lambda_j \in \mathbb{R},$$

and p is SOS if and only if there exist $\lambda_1, \cdots, \lambda_K$ such that $Q \geq 0$.

For $n = d = 2$, $z(x)$ is as defined in (7) and a basis as in (9) is:

$$N_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad N_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad N_4 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad N_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

Example 1 revisited: For $q(x_1, x_2)$ in (3), a suitable choice for Q_0 is

$$Q_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 6 \end{bmatrix}.$$

Note that $Q_0 \not\geq 0$, but $Q_0 + 6N_6 \geq 0$. Moreover, $Q_0 + 6N_6$ can be decomposed as

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 3 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 3 & 1 \end{bmatrix}$$

which explains how the SOS form (5) was obtained.

Synthesizing SOS Polynomials

With the method above we can numerically check whether a given polynomial function V satisfies (1)-(2). However, in practice, it is more important to be able to *search* for a V satisfying (1)-(2). This is accomplished by synthesizing V as a weighted sum of basis polynomials with weights left as decision variables.

This leads to the following SOS synthesis problem:

Given basis polynomials p_i, $i = 0, 1, \cdots, m$, each with degree $\leq 2d$, find parameters $a_1, \cdots, a_m$ such that $p_0 + a_1 p_1 + \cdots + a_m p_m$ is SOS.

To solve this problem, find a matrix Q_i satisfying $p_i = z^T Q_i z$ for each $i = 0, 1, \cdots, m$. Then search for $a_1, \cdots, a_m$ and $\lambda_1, \cdots, \lambda_K$ satisfying

$$Q_0 + \sum_{i=1}^{m} a_i Q_i + \sum_{j=1}^{K} \lambda_j N_j \geq 0. \tag{10}$$

This is a linear matrix inequality (LMI) and can be solved numerically with standard semidefinite program (SDP) solvers.

There are also software packages[4] that follow the procedures above to automatically convert SOS programs to LMIs, such as (10).

[4] *e.g.*, SOSOPT