Sum of Squares Programming

Establishing nonnegativity of functions is critical in nonlinear system analysis, e.g., a Lyapunov function \( V \) for \( \dot{x} = f(x) \) must satisfy

\[
V(x) > 0 \quad \forall x \neq 0 \quad (1)
\]

\[
-\nabla V(x)^T f(x) \geq 0 \quad \forall x. \quad (2)
\]

For \( f(x) = Ax \) and \( V(x) = x^T P x \), the conditions above are simple matrix inequalities:

\[ P > 0, \quad -A^T P - PA \geq 0. \]

How can we check nonnegativity when \( f \) and \( V \) are more general polynomials?

Sum of Squares (SOS) Polynomials

A monomial is a product of powers of variables (e.g., \( m(x) = x_1^2 x_2 \)) and its degree is the sum of its exponents (e.g., 3 for \( m(x) = x_1^2 x_2 \)).

A polynomial is a finite linear combination of monomials and its degree is the maximum degree of these monomials.

Example 1: The polynomial

\[
q(x_1, x_2) = x_1^2 - 2x_1 x_2^2 + 2x_1^4 + 2x_1^2 x_2 - x_1^2 x_2^2 + 6x_2^4 \quad (3)
\]

has degree 4.

Definition: A polynomial \( p \) is a sum of squares (SOS) if there exist polynomials \( g_1, \cdots, g_r \) such that

\[
p = \sum_{i=1}^r g_i^2. \quad (4)
\]

A SOS polynomial \( p(x) \) is nonnegative for all \( x \). The converse is not true: there exist nonnegative polynomials that are not SOS.

The polynomial \( q(x_1, x_2) \) in (3) is SOS because it can be rewritten as:

\[
(x_1 - x_2^2)^2 + \frac{1}{2} \left( 2x_1^2 + x_1 x_2 - 3x_2^2 \right)^2 + \frac{1}{2} \left( 3x_1 x_2 + x_2^2 \right)^2. \quad (5)
\]

You can verify the equivalence of (3) and (5) by multiplying out terms in (5) and matching them to those in (3).

How a SOS decomposition like (5) can be obtained is discussed next.
SOS Decomposition

Let \( z(x) \) be the vector of all monomials of degree \( \leq d \) in \( n \) variables\(^2\):
\[
z(x) \triangleq [1, x_1, x_2, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_n^2]^T.
\]
Then any polynomial with degree \( \leq 2d \) can be rewritten as
\[
p(x) = z(x)^T Q z(x)
\]
where \( Q \) is a symmetric matrix.

**Example 2:** Let \( p(x_1, x_2) = 2x_1^2x_2^2 \) which has degree 4. With \( n = 2 \) and \( d = 2 \),
\[
z(x) = [1, x_1, x_2, x_1^2, x_1 x_2, x_2^2]^T,
\]
and (6) holds with either
\[
Q_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \quad \text{or} \quad Q_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}.
\]

Thus, the choice of \( Q \) is not unique.

**Theorem:** A polynomial \( p \) with degree \( \leq 2d \) is SOS if and only if there exists \( Q = Q^T \geq 0 \) satisfying (6).

**Proof:** (only if) If \( p \) is SOS then, by definition, \( p = \sum_{i=1}^{r} g_i^2 \) for some polynomials \( g_i, i = 1, \ldots, r \). Write \( g_i \) as:
\[
g_i(x) = C_i z(x)
\]
where \( C_i \) is a row vector of coefficients. Then \( g_i^2 = z^T C_i^T C_i z \) and
\[
p = \sum_{i=1}^{r} g_i^2 = z^T \left( \sum_{i=1}^{r} C_i^T C_i \right) z.
\]

(if) Given \( Q = Q^T \geq 0 \) satisfying (6), decompose \( Q \) as \( Q = C^T C \) where \( C \) has as many rows as the rank of \( Q \), say \( r \). Then,
\[
Q = C^T C = \sum_{i=1}^{r} C_i^T C_i
\]
where \( C_i \) is the \( i \)th row. If we define \( g_i \) as in (8), then \( z^T Q z = \sum_{i=1}^{r} g_i^2 \). \(\square\)

Since \( Q \) is not unique, not all \( Q \) satisfying (6) will certify SOS. In Example 2 above, \( Q_1 \geq 0 \) but \( Q_2 \) is indefinite. We need to characterize the set of all \( Q \) satisfying (6) and search for a \( Q \geq 0 \) in this set.
Parameterization of all matrices $Q$ satisfying (6):

Find a particular solution $Q_0$ such that

$$p(x) = z(x)^T Q_0 z(x),$$

and find a basis of symmetric matrices $N_j$, $j=1,2,\cdots,K$, such that

$$z(x)^T N_j z(x) = 0 \quad \text{for all } x.$$  \hfill(9)

Then we can parameterize the set of all $Q$ satisfying (6) as

$$Q = Q_0 + \sum_{j=1}^{K} \lambda_j N_j \quad \lambda_j \in \mathbb{R},$$

and $p$ is SOS if and only if there exist $\lambda_1,\cdots,\lambda_K$ such that $Q \geq 0$.

For $n=d=2$, $z(x)$ is as defined in (7) and a basis as in (9) is:

$$N_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad N_4 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$N_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad N_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

Example 1 revisited: For $q(x_1,x_2)$ in (3), a suitable choice for $Q_0$ is

$$Q_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 6 \end{bmatrix}. $$
Note that $Q_0 \nless 0$, but $Q_0 + 6N_6 \geq 0$. Moreover, $Q_0 + 6N_6$ can be decomposed as
\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 2 & 1 & -3 \\
0 & 0 & 0 & 0 & 3 & 1
\end{bmatrix}^T
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1/2 & 0 \\
0 & 0 & 1/2
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 2 & 1 & -3 \\
0 & 0 & 0 & 0 & 3 & 1
\end{bmatrix}
\]
which explains how the SOS form (5) was obtained.

**Synthesizing SOS Polynomials**

With the method above we can numerically check whether a given polynomial function $V$ satisfies (1)-(2). However, in practice, it is more important to be able to search for a $V$ satisfying (1)-(2). This is accomplished by synthesizing $V$ as a weighted sum of basis polynomials with weights left as decision variables.

This leads to the following SOS synthesis problem:

*Given basis polynomials $p_i$, $i = 0, 1, \cdots, m$, each with degree $\leq 2d$, find parameters $a_1, \cdots, a_m$ such that $p_0 + a_1 p_1 + \cdots + a_m p_m$ is SOS.*

To solve this problem, find a matrix $Q_i$ satisfying $p_i = z^T Q_i z$ for each $i = 0, 1, \cdots, m$. Then search for $a_1, \cdots, a_m$ and $\lambda_1, \cdots, \lambda_K$ satisfying
\[
Q_0 + \sum_{i=1}^m a_i Q_i + \sum_{j=1}^K \lambda_j N_j \geq 0. \tag{10}
\]

This is a linear matrix inequality (LMI) and can be solved numerically with standard semidefinite program (SDP) solvers.

There are also software packages\(^4\) that follow the procedures above to automatically convert SOS programs to LMIs, such as (10).\(^4\) e.g., SOSOPT