EE C222/ME C237 - Spring'18 - Lecture 17 Notes[1]

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Review of Sum of Squares (SOS) Polynomials

Checking whether a polynomial is SOS

A polynomial p with degree $\leq 2d$ is a sum of squares if and only if there exists $Q = Q^T \geq 0$ s.t.

$$p(x) = z(x)^T Q\, z(x) \tag{1}$$

where $z(x)$ is the vector of all monomials of degree $\leq d$:

$$z(x) \triangleq [1, x_1, x_2, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_n^d]^T.$$

Find a particular solution Q_0 such that

$$p(x) = z(x)^T Q_0 z(x),$$

and find a basis of symmetric matrices N_j, $j = 1, 2, \cdots, K$, such that

$$z(x)^T N_j z(x) = 0 \quad \text{for all } x. \tag{2}$$

Then p is SOS if and only if there exist reals $\lambda_1, \cdots, \lambda_K$ such that

$$Q = Q_0 + \sum_{j=1}^{K} \lambda_j N_j \geq 0. \tag{3}$$

This is a linear matrix inequality (LMI) and can be solved numerically with standard semidefinite program (SDP) solvers.

Synthesizing SOS Polynomials

Given p_i, $i = 0, 1, \cdots, m$, each with degree $\leq 2d$, find reals $a_1, \cdots, a_m$ s.t. $p_0 + a_1 p_1 + \cdots + a_m p_m$ is SOS.

Find a particular Q_i satisfying $p_i = z^T Q_i z$ for each $i = 0, 1, \cdots, m$. Then search for $a_1, \cdots, a_m$ and $\lambda_1, \cdots, \lambda_K$ satisfying the LMI

$$Q_0 + \sum_{i=1}^{m} a_i Q_i + \sum_{j=1}^{K} \lambda_j N_j \geq 0. \tag{4}$$

Applications

Searching for a Lyapunov Function

Given $\dot{x} = f(x)$, $f(0) = 0$, where f is a vector of polynomials, search for a Lyapunov function of the form

$$V(x) = p_0(x) + a_1 p_1(x) + \cdots + a_m p_m(x) \tag{5}$$

where p_i, $i = 0, 1, \cdots, m$ are basis polynomials selected ahead of time, and a_i, $i = 1, \cdots, m$ are weights to be determined.

To ensure V is positive definite, pick a positive definite polynomial ℓ (e.g., $\ell(x) = \varepsilon x^T x$ for some small ε) and impose the constraint:

$$V(x) - \ell(x) \text{ is SOS.} \tag{6}$$

To ensure $\nabla V(x)^T f(x)$ is negative semidef., impose the constraint:

$$-\nabla V(x)^T f(x) \text{ is SOS.} \tag{7}$$

Constraints (6) and (7) can be brought to the LMI form (4) and feasible a_i, $i = 1, \cdots, m$ can be determined numerically (if they exist).

Overapproximating Reachable Sets

Recall from Lecture 15 that

$$R_T \triangleq \left\{ x(T) \mid \dot{x} = f(x,u),\ x(0) = 0,\ \int_0^T u^T(t)u(t)dt \leq 1 \right\} \tag{8}$$

defines the reachable set from $x(0) = 0$ under unit energy inputs and, if we can find a positive definite V such that

$$\nabla V(x)^T f(x,u) \leq u^T u, \tag{9}$$

then we can overapproximate R_T by:

$$R_T \subset \{x : V(x) \leq 1\}.$$

This follows because, from (9),

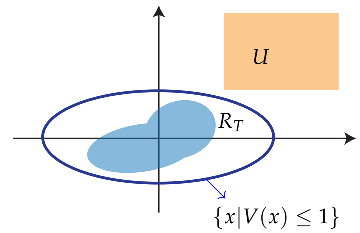
$$\begin{aligned} \frac{d}{dt} V(x(t)) \leq u^T u \ &\Rightarrow\ V(x(T)) - V(x(0)) \leq \int_0^T u^T(t)u(t)dt \leq 1 \\ &\Rightarrow\ V(x(T)) \leq 1. \end{aligned}$$

If $f(x,u)$ is a vector of polynomials in x and u, we can search for a polynomial V of the form (5), and encode (9) with the constraint:

$$-\nabla V(x)^T f(x,u) + u^T u \text{ is SOS in } x \text{ and } u. \tag{10}$$

This can then be combined with (6) and brought to the LMI form (4).

Certifying Safety



If unsafe set U does not intersect the overapproximation above, then it can't intersect the actual reachable set. Thus, we can certify safety by proving the implication:

$$x \in U \quad \Rightarrow \quad V(x) \geq 1 + \varepsilon \tag{11}$$

for some $\varepsilon > 0$.

Suppose the unsafe set can be expressed as

$$U = \{x : q_i(x) \geq 0, \, i = 1, \cdots, p\}$$

where q_i are polynomials. Then we can encode (11) with the constraints:

$$V(x) - (1 + \varepsilon) - \sum_{i=1}^{p} s_i(x) q_i(x) \text{ is SOS} \tag{12}$$

$$s_i(x), \; i = 1, \cdots, p \text{ are SOS.} \tag{13}$$

We can parameterize the search space for s_i as we did for V in (5), and combine (6), (10), (12)-(13) into a LMI.

Above we implicitly used a generalization of the S-procedure from Lecture 15. Specifically, to prove that

$$q_0(x) \geq 0 \quad \text{whenever} \quad q_i(x) \geq 0, \; i = 1, 2, \ldots, p$$

we look for nonnegative functions $s_1, s_2, \ldots, s_p$ (rather than constants as in Lecture 15) such that

$$q_0(x) - \sum_{i=1}^{p} s_i(x) q_i(x) \geq 0.$$

Underapproximating the Region of Attraction

Given system $\dot{x} = f(x)$ with asymptotically stable equilibrium at the origin $x = 0$, the region of attraction, denoted R_A, is the set of initial conditions from which the trajectories converge to the origin.

Recall from Lecture 10 that, if V is positive definite and

$$\nabla V(x)^T f(x) < 0 \quad \text{whenever } x \neq 0 \text{ and } V(x) \leq \gamma, \tag{14}$$

then $\Omega_\gamma \triangleq \{x : V(x) \leq \gamma\} \subset R_A$.

Let ℓ be a positive definite polynomial. If there exists a SOS polynomial s such that

$$-[\ell(x) + \nabla V(x)^T f(x)] - s(x)[\gamma - V(x)] \text{ is SOS,} \tag{15}$$

then $V(x) \leq \gamma$ implies $\nabla V(x)^T f(x) \leq -\ell(x)$ as stipulated in (14).

To obtain a LMI from (15), one option is to fix the Lyapunov function[2] V and to parameterize the search space for s. We can further maximize γ subject to (15) by incrementing γ until the the resulting LMI is infeasible.

[2] choose, *e.g.*, a quadratic Lyapunov function for the linearized model at $x = 0$

Alternatively s can be fixed and V parameterized. If we parameterize both s and V, however, (15) is no longer affine in the parameters because the term $s(x)V(x)$ contains the products of these parameters.

Below is a procedure that alternates between first fixing V, varying s, and next fixing s, varying V. When a new V is obtained, however, the shape of the level set changes and it may be ambiguous whether the new one is bigger. To remove this ambiguity we define a "shape function" p and use its level sets to judge the size of the region of attraction estimate.

Step 1: Let $V_0(x)$ be an initial choice for a Lyapunov function, *e.g.*, a quadratic function for the linearized model at the origin. Find

$$\gamma^* := \max \gamma \quad \text{s.t.} \quad \nabla V_0(x)^T f(x) < 0 \text{ whenever } x \neq 0 \text{ and } V_0(x) \leq \gamma.$$

To satisfy the constraint look for a SOS multiplier $s_1(x)$ that satisfies

$$-[\ell(x) + \nabla V_0(x)^T f(x)] - s_1(x)[\gamma - V_0(x)] \text{ is SOS}$$

where ℓ is positive definite, *e.g.*, $\ell(x) := \epsilon(x_1^2 + x_2^2)$ for some $\epsilon > 0$.

Step 2: Let $p(x)$ be some fixed, positive definite convex polynomial (*e.g.*, $p(x) = x_1^2 + x_2^2$), and let $V_0(x)$ and γ^* be as in Step 1. Find

$$\beta^* := \max \beta \quad \text{s.t.} \quad V_0(x) \leq \gamma^* \text{ whenever } p(x) \leq \beta.$$

To satisfy the constraint look for a SOS multiplier $s_2(x)$ such that

$$[\gamma^* - V_0(x)] - s_2(x)[\beta - p(x)] \text{ is SOS.}$$

This means that $\{x : p(x) \leq \beta\}$ is contained in $\{x : V_0(x) \leq \gamma^*\}$.

Step 3: Given $\gamma^*, s_1(x)$ from Step 1 and $p(x), s_2(x)$ from Step 2, search for $V(x)$ to solve:

$$\begin{aligned}
\max_{\substack{\beta>0 \\ \text{4th-order } V(x)}} \quad & \beta \\
\text{subject to} \quad & V(x) - \ell(x) \text{ is SOS} \\
& -[\ell(x) + \nabla V(x)^T f(x)] - s_1(x)[\gamma^* - V(x)] \text{ is SOS} \\
& [\gamma^* - V(x)] - s_2(x)[\beta - p(x)] \text{ is SOS.}
\end{aligned}$$

The first constraint ensures V is positive definite. The second implies that the level set $\{x : V(x) \leq \gamma^*\}$ is invariant, hence a valid approximation for the region of attraction. The third constraint and the maximization of β ensure that V is selected such that the level set $\{x : V(x) \leq \gamma^*\}$ is as large as possible, as measured by function p.

To proceed, replace $V_0(x)$ in Step 1 with the function $V(x)$ from Step 3, and repeat the steps above for several iterations, until the change in β^* in Step 2 is sufficiently small. The final approximation of the ROA is the set where $V(x) \leq \gamma^*$.