

EE222 - Spring'18 - Lecture 19 Notes¹

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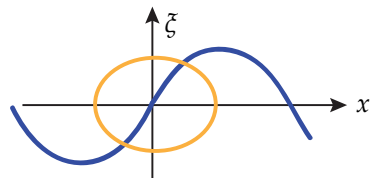
Feedback Linearization (continued)

Nonlinear Changes of Variables

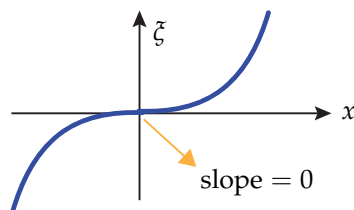
$T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called a *diffeomorphism* if its inverse T^{-1} exists, and both T and T^{-1} are continuously differentiable (C^1).

Examples:

1. $\zeta = Tx$ is a diffeomorphism if T is a nonsingular matrix
2. $\zeta = \sin x$ is a local diffeomorphism around $x = 0$, but not global



3. $\zeta = x^3$ is not a diffeomorphism because $T^{-1}(\cdot)$ is not C^1 at $\zeta = 0$



How to check if $\zeta = T(x)$ is a local diffeomorphism?

Implicit Function Theorem

Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is C^1 and there exists $x_0 \in \mathbb{R}^n$, $\zeta_0 \in \mathbb{R}^m$ such that

$$f(x_0, \zeta_0) = 0.$$

If $\frac{\partial f}{\partial x}(x_0, \zeta_0)$ is nonsingular, then in a neighborhood of (x_0, ζ_0) ,

$$f(x, \zeta) = 0$$

has a unique solution $x = g(\zeta)$ where g is C^1 at $\zeta = \zeta_0$.

Corollary: Let $f(x, \zeta) = T(x) - \zeta$. If $\frac{\partial T}{\partial x}$ is nonsingular at x_0 , then $T(\cdot)$ is a local diffeomorphism around x_0 .

A "Normal Form" that Explicitly Displays the Zero Dynamics

Theorem: If $\dot{x} = f(x) + g(x)u$, $y = h(x)$ has a well-defined relative degree $r \leq n$, then there exist a diffeomorphism $T : x \rightarrow \begin{bmatrix} z \\ \zeta \end{bmatrix}$, $z \in \mathbb{R}^{n-r}$, $\zeta \in \mathbb{R}^r$, that transforms the system to the form:

$$\begin{aligned} \dot{z} &= f_0(z, \zeta) \\ \dot{\zeta}_1 &= \zeta_2 \\ &\vdots \\ \dot{\zeta}_r &= b(z, \zeta) + a(z, \zeta)u, \quad y = \zeta_1. \end{aligned} \tag{1}$$

In particular, $\dot{z} = f_0(z, 0)$ represents the zero dynamics. \square

To obtain this form, let $\zeta = [h(x) \quad L_f h(x) \quad \dots \quad L_f^{r-1} h(x)]^T$, and find $n - r$ independent variables z such that \dot{z} does not contain u .

Note that the terms $b(z, \zeta)$ and $a(z, \zeta)$ correspond to $L_f^r(x)$ and $L_g L_f^{r-1} h(x)$ in the original coordinates.

Example:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha x_3 + u \\ \dot{x}_3 &= \beta x_3 - u \\ y &= x_1. \end{aligned}$$

Let $\zeta_1 = x_1$, $\zeta_2 = x_2$, and note that $z = x_2 + x_3$ is independent of ζ_1, ζ_2 , and \dot{z} does not contain u . Thus, the normal form is:

$$\begin{aligned} \dot{z} &= (\alpha + \beta)x_3 = (\alpha + \beta)z - (\alpha + \beta)\zeta_2 \\ \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= \alpha x_3 + u = \alpha z - \alpha \zeta_2 + u. \end{aligned}$$

I/O Linearizing Controller in the new coordinates (1):

$$u = \frac{1}{a(z, \zeta)} \left(-b(z, \zeta) + v \right) \tag{2}$$

$$v = -k_1 \zeta_1 \dots - k_r \zeta_r \tag{3}$$

where k_1, \dots, k_r are such that all roots of $s^r + k_r s^{r-1} + \dots + k_2 s + k_1$ have negative real parts.

Theorem: If $z = 0$ is locally exponentially stable for the zero dynamics $\dot{z} = f_0(z, 0)$, then (2)–(3) locally exponentially stabilizes $x = 0$.

Proof: Closed-loop system:

$$\begin{aligned} \dot{z} &= f_0(z, \zeta) \\ \dot{\zeta} &= A\zeta \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ & & & \ddots \\ & & & & 1 \\ -k_1 & -k_2 & -k_3 & \dots & -k_r \end{bmatrix}$$

is Hurwitz. The Jacobian linearization at $(z, \zeta) = 0$ is:

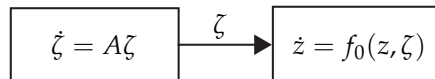
$$J = \begin{bmatrix} \frac{\partial f_0}{\partial z}(0,0) & \frac{\partial f_0}{\partial \zeta}(0,0) \\ 0 & A \end{bmatrix}$$

where $\frac{\partial f_0}{\partial z}(0,0)$ is Hurwitz since $\dot{z} = f_0(z,0)$ is exponentially stable by the proposition in Lecture 11, page 1. Since A is also Hurwitz, all eigenvalues of J have negative real parts \Rightarrow exponential stability.

Global asymptotic stability can be guaranteed with additional assumptions on the zero dynamics, such as ISS of

$$\dot{z} = f_0(z, \zeta)$$

with respect to the input ζ :



Example: $\dot{z} = -z + z^2\zeta, \quad \dot{\zeta} = -k\zeta$

Note: the z subsystem is not ISS

$(z, \zeta) = 0$ is locally exponentially stable, but not globally: solutions escape in finite time for large $z(0)$.

I/O Linearizing Controller for Tracking

For the output $y(t)$ to track a reference signal² $y_d(t)$, replace (3) with:

² assumed to be r times differentiable

$$v = -k_1(\zeta_1 - y_d(t)) - k_2(\zeta_2 - \dot{y}_d(t)) \cdots - k_r(\zeta_r - y_d^{(r-1)}(t)) + y_d^{(r)}(t)$$

Let $e_1 \triangleq \zeta_1 - y_d(t), e_2 \triangleq \zeta_2 - \dot{y}_d(t), \dots, e_r \triangleq \zeta_r - y_d^{(r-1)}(t)$. Then:

$$\left. \begin{array}{l} \dot{e}_1 = e_2 \\ \dot{e}_2 = e_3 \\ \vdots \\ \dot{e}_r = v - y_d^{(r)}(t) = -k_1 e_1 - \cdots - k_r e_r \end{array} \right\} \dot{e} = Ae.$$

Thus $e(t) \rightarrow 0$, that is $y(t) - y_d(t) \rightarrow 0$.

If $y_d(t)$ and its derivatives are bounded, then $\zeta(t)$ is bounded. If the zero dynamics $\dot{z} = f_0(z, \zeta)$ is ISS with respect to ζ , then $z(t)$ is also bounded. Thus, all internal signals are bounded.