EE C222/ME C237 - Spring'18 - Lecture 21 Notes[1]

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Feedback Linearization Continued

Recall "strict feedback systems" discussed in Lecture 13:

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\
\dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)x_4 \\
&\vdots \\
\dot{x}_n &= f_n(x) + g_n(x)u.
\end{aligned} \tag{1}
$$

Such systems are feedback linearizable when $g_i(x_1, \ldots, x_i) \neq 0$ near the origin, $i = 1, 2, \cdots, n$, because the relative degree is n with the choice of output $y = h(x) = x_1$:

$$
y^{(n)} = L_f^n h(x) + \underbrace{g_1(x_1)g_2(x_1, x_2)\cdots g_n(x)}_{L_g L_f^{n-1}h(x) \neq 0} u.
$$

Feedback linearizability is lost when $g_i(0) = 0$ for some i; however, backstepping may be applicable as the following example illustrates:

Example 1:

$$
\begin{aligned}
\dot{x}_1 &= x_1^2 x_2 \\
\dot{x}_2 &= u.
\end{aligned}
$$

Treat x_2 as virtual control and let $\alpha_1(x_1) = -x_1$ which stabilizes the x_1-subsystem, as verified with Lyapunov function $V_1(x_1) = \frac{1}{2}x_1^2$. Then $z_2 := x_2 - \alpha_1(x_1)$ satisfies $\dot{z}_2 = u - \dot{\alpha}_1$, and

$$
u = \dot{\alpha}_1 - \frac{\partial V_1}{\partial x_1}x_1^2 - k_2 z_2 = -x_1^2 x_2 - x_1^3 - k_2(x_2 + x_1)
$$

achieves global asymptotic stability:

$$
V = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2 \quad \Rightarrow \quad \dot{V} = -{x_1}^4 - k_2 z_2^2.
$$

In contrast the system is not feedback linearizable, because condition (C1) in the theorem for feedback linearizability (Lecture 20, p.4) fails. To see this note that

$$
f(x) = \begin{bmatrix} x_1^2 x_2 \\ 0 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathrm{ad}_f\, g(x) = [f, g](x) = \begin{bmatrix} -x_1^2 \\ 0 \end{bmatrix},
$$

thus, with $n = 2$ and $x_0 = 0$,

$$[g(x_0) \;\; \mathrm{ad}_f\, g(x_0) \;\; \ldots \;\; \mathrm{ad}_f^{n-1} g(x_0)] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

which is rank deficient.

Multi-Input Multi-Output Systems

Consider now a MIMO system with m inputs and m outputs:

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i \\ y_i &= h_i(x), \quad i = 1, \cdots, m. \end{aligned} \tag{2}$$

Let r_i denote the number of times we need to differentiate y_i to hit at least one input. Then,

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_m^{(r_m)} \end{bmatrix} = \underbrace{\begin{bmatrix} L_f^{r_1} h_1(x) \\ \vdots \\ L_f^{r_m} h_m(x) \end{bmatrix}}_{=:\, B(x)} + \underbrace{\begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix}}_{=:\, A(x)} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}.$$

If $A(x)$ is nonsingular, then the feedback law

$$u = A(x)^{-1}(-B(x) + v)$$

input/output linearizes the system, creating m decoupled chains of integrators:

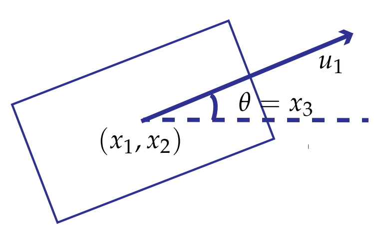
$$y_i^{(r_i)} = v_i, \quad i = 1, \ldots, m.$$

We say that the system has *vector relative degree* $\{r_1, \cdots, r_m\}$ if the matrix $A(x)$ defined above is nonsingular.

Example 2: The kinematic model of a unicycle, depicted below, is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2,$$

where u_1 is the speed and u_2 is the angular velocity.



Let $y_1 = x_1$ and $y_2 = x_2$, and note that

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \end{bmatrix}}_{=: A(x)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Since $A(x)$ is singular, the system does not have a well-defined vector relative degree. □

The notion of zero dynamics and the normal form can be extended to MIMO systems[2]. If the system has vector relative degree $\{r_1, \cdots, r_m\}$, then $r := r_1 + \cdots + r_m \leq n$ and

[2] see, *e.g.*, Sastry, Section 9.3

$$\zeta := [h_1(x)\ L_f h_1(x) \cdots L_f^{r_1-1} h_1(x)\ \ \cdots\ \ h_m(x)\ L_f h_m(x) \cdots L_f^{r_m-1} h_m(x)]^T$$

defines a partial set of coordinates. As in normal form discussed in Lecture 19, one can find $n - r$ additional functions $z_1(x), \cdots, z_{n-r}(x)$ so that $x \mapsto (z, \zeta)$ is a complete coordinate transformation.

Full-state feedback linearization amounts to finding m output functions $h_1, \cdots, h_m$ such that the system has vector relative degree $\{r_1, \cdots, r_m\}$ with $r_1 + \cdots + r_m = n$. Necessary and sufficient conditions for the existence of such functions, analogous to those in Lecture 20 for SISO systems, are available[3].

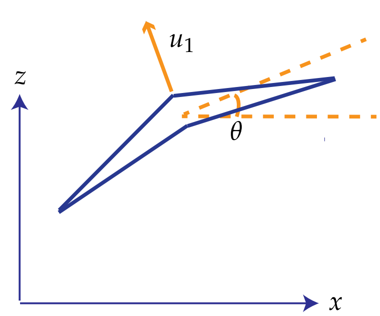
[3] see, *e.g.*, Sastry, Proposition 9.16

Example 3: Consider the following model of a *planar vertical take-off and landing* (PVTOL) aircraft[4]

[4] Sastry, Section 10.4.2

$$\begin{aligned} \ddot{x} &= -\sin(\theta)u_1 + \mu\cos(\theta)u_2 \\ \ddot{z} &= \cos(\theta)u_1 + \mu\sin(\theta)u_2 - 1 \\ \ddot{\theta} &= u_2, \end{aligned}$$

where μ is a constant that accounts for the coupling between the rolling moment and translational acceleration, and -1 in the second equation is the gravitational acceleration, normalized to unity by appropriately scaling the variables.



If we take x and z as the two outputs we get

$$\begin{bmatrix} \ddot{x} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \underbrace{\begin{bmatrix} -\sin\theta & \mu\cos\theta \\ \cos\theta & \mu\sin\theta \end{bmatrix}}_{A(\theta)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $A(\theta)$ is invertible when $\mu \neq 0$:

$$A^{-1}(\theta) = \begin{bmatrix} -\sin\theta & \cos\theta \\ \frac{1}{\mu}\cos\theta & \frac{1}{\mu}\sin\theta \end{bmatrix}.$$

Thus the systems has vector relative degree $\{2,2\}$ when $\mu \neq 0$, and the input/output linearizing controller is

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta \\ \frac{1}{\mu}\cos\theta & \frac{1}{\mu}\sin\theta \end{bmatrix} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right).$$

The zero dynamics is obtained by substituting $u_2^* = \frac{1}{\mu}\sin\theta$, needed to maintain z at a constant value and $\dot{z}$ at zero, in the dynamical equation for θ:

$$\ddot{\theta} = \frac{1}{\mu}\sin\theta.$$

The system is nonminimum phase for $\mu > 0$, since $\theta = 0$ is unstable.

Drift-Free Systems

Suppose $f(x) = 0$ for all x in (2). Such system are called *drift-free* and encompass linear systems of the form

$$\dot{x} = Bu, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m.$$

Assuming the columns of the $n \times m$ matrix B are linearly independent, we can find $n - m$ row vectors T_i, $i = 1, \cdots, n-m$, such that

$$T_i B = 0.$$

This means that $\phi_i(x) := T_i x$ satisfies

$$\frac{d}{dt}\phi_i(x(t)) = 0 \quad \Rightarrow \quad \phi_i(x(t)) = \phi_i(x(0)) \tag{3}$$

regardless of the control inputs. Since there are $n - m$ such constraints, controllability is not possible in drift-free linear systems with fewer control inputs than the state dimension ($m < n$).

The Frobenius Theorem (Lecture 20) implies that constraints of the form (3), called *holonomic constraints*, also exist for nonlinear drift-free systems

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i \tag{4}$$

when the distribution $\Delta = \text{span}\{g_1, \cdots, g_m\}$ is nonsingular and involutive.

When Δ is non-involutive, however, controllability may be possible with $m < n$ – another essentially nonlinear phenomenon.

Indeed, *Chow's Theorem* states that (4) is controllable if the *involutive closure*[5] of $\Delta = \text{span}\{g_1, \cdots, g_m\}$ has dimension n. This condition means that the Lie brackets of $g_1, \cdots, g_m$ span new dimensions that are not already spanned by these basis vector fields. Drift-free systems satisfying Chow's Theorem are called *nonholonomic*.

[5] the smallest involutive distribution that containts Δ

<u>Example 4:</u> Recall the unicycle model discussed in Example 2, where

$$g_1(x) = \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad [g_1, g_2](x) = \begin{bmatrix} -\sin x_3 \\ \cos x_3 \\ 0 \end{bmatrix}.$$

$\Delta = \text{span}\{g_1, g_2\}$ is non-involutive, as $[g_1, g_2]$ generates a new direction. Taken together, g_1, g_2, and $[g_1, g_2]$ span the entire three-dimensional space at each point x; therefore, the system is controllable by Chow's Theorem. This conclusion sheds light on how parallel parking is possible despite lack of sideways actuation. □

To present an interpretation of the Lie bracket $[g_1, g_2]$, we let $\Phi_t^{g_i}(x_0)$ denote the solution of the system $\dot{x} = g_i(x)$ at time t from initial condition x_0. Then it can be shown that

$$\Phi_t^{-g_2}(\Phi_t^{-g_1}(\Phi_t^{g_2}(\Phi_t^{g_1}(x_0)))) = t^2[g_1, g_2](x_0) + \mathcal{O}(t^3),$$

which suggests that motion in the direction of the Lie bracket $[g_1, g_2]$ can be generated by alternating actuation of the two inputs u_1 and u_2 with positive and negative signs, as one does in parallel parking.