EE C222/ME C237 - Spring'18 - Lecture 22 Notes[1]

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Finite Time Convergence

Systems with Lipschitz continuous dynamics converge to equilibrium points no faster than exponentially (Homework 10, Problem 1). Finite-time convergence is thus possible only with non-Lipschitz or discontinuous dynamics, as illustrated in the following examples.

Example 1: Consider the system $\dot{x} = -x^{1/3}$, where the right-hand side is not Lipschitz. We rearrange the differential equation as $x^{-1/3}\dot{x} = \frac{3}{2}\frac{d}{dt}x^{2/3} = -1$ for $x \neq 0$, and obtain the solution

$$x(t)^{2/3} = x(0)^{2/3} - \frac{2t}{3}$$

which holds until $x(t)$ reaches 0 at $t = \frac{3}{2}x(0)^{2/3}$.

Example 2: Consider the system $\dot{x} = -\text{sgn}(x)$ where

$$\text{sgn}(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

The solution is $x(t) = x(0) - t$ when $x(0) > 0$ and $x(t) = x(0) + t$ when $x(0) < 0$, until $x(t)$ reaches zero at $t = |x(0)|$ in each case.

The following proposition allows us to conclude finite time convergence from a Lyapunov function.

Proposition: Consider the system $\dot{x} = f(t,x)$, $f(t,0) = 0\ \forall t$. If there exists a positive definite, continuously differentiable and radially unbounded function $V : \mathbb{R}^n \mapsto \mathbb{R}$, and constants $c > 0$ and $\alpha \in (0,1)$ such that, for all t and x,

$$\dot{V}(x) := \nabla V(x)^T f(t,x) \leq -cV(x)^{\alpha}$$

then all trajectories converge to the origin in finite time.

The proof follows by defining $w(t) := V(x(t))$, which satisfies the differential inequality $\dot{w}(t) \leq -cw(t)^{\alpha}$. Finite time convergence of $w(t)$ and, thus of $x(t)$, can then be argued by rearranging and solving the differential equation $\dot{\bar{w}} = -c\bar{w}^{\alpha}$, $\bar{w}(0) = w(0)$, as in Example 1 above, and noting that $w(t) \leq \bar{w}(t)$.

As an illustration, in Example 2 above $V = \frac{1}{2}x^2$ yields

$$\dot{V} = -x\,\text{sgn}(x) = -|x| = -\sqrt{2V},$$

which satisfies the proposition above with $c = \sqrt{2}$ and $\alpha = 1/2$.

Example 3: Consider the control system

$$\dot{x} = u + \delta(x), \quad x \in \mathbb{R}, u \in \mathbb{R},$$

where $\delta(x)$ is unknown, but an upper bound $\rho(x)$ is available:

$$|\delta(x)| \leq \rho(x).$$

To stabilize the origin despite the unknown $\delta(x)$ we can apply

$$u = -(\rho(x) + \rho_0)\text{sgn}(x),$$

where $\rho_0 > 0$ is a constant. Then $V = \frac{1}{2}x^2$ gives

$$\begin{aligned}
\dot{V} &= -(\rho(x) + \rho_0)|x| + x\delta(x) \\
&\leq -(\rho(x) + \rho_0)|x| + |x||\delta(x)| \\
&= -\rho_0|x| - (\rho(x) - |\delta(x)|)|x| \\
&\leq -\rho_0|x| = -\rho_0\sqrt{2V}.
\end{aligned}$$

This implies that, in addition to dominating the uncertain term $\delta(x)$, we achieve finite time stability of $x = 0$.

Sliding Mode Control

Example 3 demonstrated the ability of a discontinuous controller to dominate uncertain terms. Sliding mode control extends this idea to higher order systems, as illustrated in the following example.

Example 4: Consider the second order system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= h(x) + g(x)u
\end{aligned} \tag{1}$$

where $g(x) \geq g_0 > 0 \ \forall x$. If we can drive the trajectories to the surface

$$s := x_2 + a_1x_1 = 0,$$

where $a_1 > 0$ is a design parameter, then x_1 is governed by $\dot{x}_1 = -a_1x_1$ on this surface and converges to zero along with $x_2 = -a_1x_1$. To ensure $s(t) \to 0$ note that

$$\dot{s} = a_1x_2 + h(x) + g(x)u, \tag{2}$$

and let ρ be a function such that

$$\frac{|a_1x_2 + h(x)|}{g(x)} \leq \rho(x). \tag{3}$$

Then apply the controller

$$u = -(\rho(x) + \rho_0)\mathrm{sgn}(s), \ \rho_0 > 0, \tag{4}$$

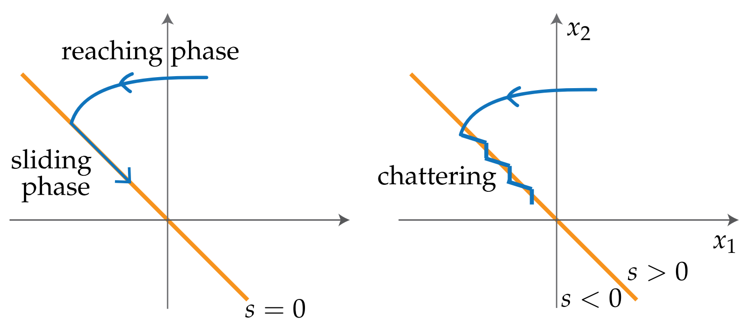
and note that $V = \frac{1}{2}s^2$ satisfies

$$\begin{aligned} \dot{V} &= s[a_1x_2 + h(x) - g(x)(\rho(x) + \rho_0)\mathrm{sgn}(s)] \\ &\leq |a_1x_2 + h(x)||s| - g(x)(\rho(x) + \rho_0)|s| \\ &\leq \underbrace{(|a_1x_2 + h(x)| - g(x)\rho(x))}_{\leq 0 \text{ by } (3)}|s| - g_0\rho_0|s|. \end{aligned}$$

Thus, $\dot{V} \leq -g_0\rho_0|s| = -\sqrt{2}g_0\rho_0V^{1/2}$, and the proposition on page 1 implies $s(t) \to 0$ in finite time.

An advantage of the controller (4) is that it does not require exact knowledge of h and g; it relies only on the upper bound (3).

The closed-loop system evolves in two phases. In the *reaching* phase the controller forces the trajectories to the surface $s = 0$ in finite time. In the *sliding* phase the trajectories slide on this surface to the origin.



In practice delays in switching lead to "chattering" around the sliding surface, as illustrated in the figure above (right).

To mitigate chattering one idea is to divide the control into a continuous part for the nominal dynamical model and a discontinuous part for the remaining uncertain terms. With this approach the magnitude of the discontinuity is reduced and, thus, chattering is less severe.

Example 4 revisited: Let $\hat{h}$ and $\hat{g} > 0$ denote nominal models for h and g. Define

$$\delta(x) := h(x) - \hat{h}(x).$$

and rewrite (2) as

$$\dot{s} = a_1x_2 + \hat{h}(x) + \delta(x) + g(x)u.$$

Then we can attempt to cancel the first two, known terms with

$$u = -\frac{a_1x_2 + \hat{h}(x)}{\hat{g}(x)} + v, \tag{5}$$

where v is left to be designed. Because the cancelation is inexact when $\hat{g} \neq g$, this results in

$$\dot{s} = \underbrace{\left(1 - \frac{g(x)}{\hat{g}(x)}\right)(a_1 x_2 + \hat{h}(x)) + \delta(x)}_{=:\,\Delta(x)} + g(x)v,$$

and the task for v is to dominate the combined uncertain term $\Delta(x)$. This is accomplished with the choice

$$v = -(r(x) + r_0)\mathrm{sgn}(s), \quad r_0 > 0, \tag{6}$$

where r is a function satisfying

$$\frac{|\Delta(x)|}{g(x)} \leq r(x). \tag{7}$$

The finite time convergence of $s(t)$ to zero follows from a Lyapunov analysis similar to that in Example 4 above.

The advantage of the control (5)-(6) is that the continuous part (5) accounts for the nominal term $\hat{h}$, save for the inexact cancelation when $\hat{g} \neq g$. Thus, the magnitude of r in the discontinuous term (6) can be significantly smaller than ρ in (4), leading to reduced chattering.

Example 5: For a specific illustration of the control design (5)-(6), consider the model

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \theta x_1^2 + u, \end{aligned}$$

where θ is an uncertain parameter in the interval $[0.9, 1.1]$. This model is of the form (1) with $h(x) = \theta x_1^2$ and $g(x) = 1$. We let $\hat{h}(x) = x_1^2$, and $\hat{g} = g = 1$, since the latter is perfectly known. Thus

$$\delta(x) := h(x) - \hat{h}(x) = (\theta - 1)x_1^2$$

where $|\theta - 1| \leq 0.1$, and we can take $r(x) = 0.1x_1^2$ to satisfy (7). The controller (5)-(6) is then

$$u = -a_1 x_2 - x_1^2 - (0.1x_1^2 + r_0)\mathrm{sgn}(a_1 x_1 + x_2), \quad a_1 > 0,\ r_0 > 0.$$

Note that the magnitude of the sgn function is diminished relative to the purely discontinuous controller (4) where ρ must satisfy (3), *e.g.*,

$$\rho(x) = a_1|x_2| + 1.1x_1^2.$$