EE C222/ME C237 - Spring'18 - Lecture 23 Notes[1]

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Sliding Mode Control Continued

We generalize the sliding mode control examples of the last lecture to the class of systems

$$\begin{aligned} \dot{\eta} &= f_a(\eta,\omega) \\ \dot{\omega} &= f_b(\eta,\omega) + \delta(\eta,\omega) + G(\eta,\omega)u, \end{aligned} \tag{1}$$

where $\omega \in \mathbb{R}^p$, $u \in \mathbb{R}^p$, $\eta \in \mathbb{R}^{n-p}$. The uncertain terms are $\delta(\eta,\omega)$ and the $p \times p$ matrix $G(\eta,\omega)$, assumed to be diagonal with entries

$$g_i(\eta,\omega) \geq g_0 > 0, \quad i = 1, \cdots, p.$$

Let $\phi(\eta)$ be a virtual control law for ω that stabilizes the origin of the η-subsystem, $\dot{\eta} = f_a(\eta,\phi(\eta))$. To drive the trajectories to the sliding surface $\omega = \phi(\eta)$, we note that $s := \omega - \phi(\eta) \in \mathbb{R}^p$ satisfies

$$\dot{s} = f_b(\eta,\omega) - \frac{\partial\phi(\eta)}{\partial\eta} f_a(\eta,\omega) + \delta(\eta,\omega) + G(\eta,\omega)u,$$

and let

$$u = -\hat{G}^{-1}(\eta,\omega)\left[f_b(\eta,\omega) - \frac{\partial\phi(\eta)}{\partial\eta} f_a(\eta,\omega)\right] + v$$

where $\hat{G}(\eta,\omega)$ is a nominal model for $G(\eta,\omega)$, and v is to be designed. Then,

$$\dot{s} = \underbrace{(I - G\hat{G})\left[f_b(\eta,\omega) - \frac{\partial\phi(\eta)}{\partial\eta} f_a(\eta,\omega)\right] + \delta(\eta,\omega)}_{=:\,\Delta(\eta,\omega)} + G(\eta,\omega)v,$$

which means that the ith entry of s satisfies

$$\dot{s}_i = \Delta_i(\eta,\omega) + g_i(\eta,\omega)v_i, \quad i = 1, \cdots, p.$$

We let

$$v_i = -(\rho_i(\eta,\omega) + \rho_0)\operatorname{sgn}(s_i), \quad \rho_0 > 0, \tag{2}$$

where $\rho_i(\eta,\omega)$ is a function such that

$$\frac{|\Delta_i(\eta,\omega)|}{g_i(\eta,\omega)} \leq \rho_i(\eta,\omega).$$

Then the Lyapunov function $V_i = \frac{1}{2}s_i^2$ satisfies $\dot{V}_i \leq -\sqrt{2}\rho_0 g_0 V^{1/2}$, which guarantees finite time convergence of s_i to 0, as discussed in Lecture 22.

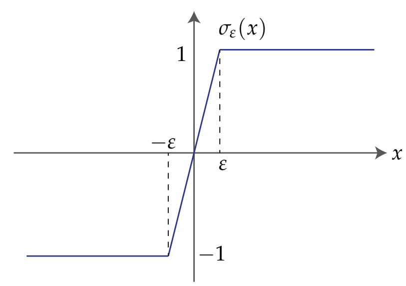
Thus the trajectories reach the sliding surface $\omega = \phi(\eta)$ in finite time and, if the subsystem $\dot{\eta} = f_a(\eta, \phi(\eta) + s)$ is ISS with respect to s, then η remains bounded during the reaching phase and converges to zero asymptotically during the sliding phase.

Continuous Approximation of Sliding Mode Control

To avoid the chattering phenomenon discussed in the previous lecture, we can employ the continuous function

$$\sigma_\varepsilon(x) := \begin{cases} x/\varepsilon & \text{when } x \in [-\varepsilon, \varepsilon] \\ \text{sgn}(x) & \text{otherwise,} \end{cases}$$

which approximates $\text{sgn}(\cdot)$ when $\varepsilon > 0$ is a small constant.



If we implement (2) above with $\sigma_\varepsilon(s_i)$ instead of $\text{sgn}(s_i)$, the Lyapunov analysis is unchanged when $|s_i| \geq \varepsilon$, where the two functions are identical. Thus, $|s_i| \geq \varepsilon$ implies $\dot{V}_i \leq -\sqrt{2}\rho_0 g_0 V^{1/2} < 0$, from which we conclude that s_i reaches the interval $[-\varepsilon, \varepsilon]$ in finite time and remains in it thereafter. Likewise, if $\dot{\eta} = f_a(\eta, \phi(\eta) + s)$ is ISS with respect to s, then η converges to a residual set around $\eta = 0$ whose size shrinks as $\varepsilon \to 0$.

Therefore, the continuous approximation eliminates chattering, but guarantees convergence to a small set around the origin rather than to the origin.

Example: For the system

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= \theta x_1^2 + u, \quad |\theta| \leq 2, \end{aligned}$$

the virtual control $\phi(x_1) = -x_1^2$ and the variable $s := x_2 - \phi(x_1) = x_2 + x_1^2$ result in

$$\dot{x}_1 = -x_1^3 + x_1 s,$$

which is ISS with respect to s. To drive s to zero we note that

$$\dot{s} = 2x_1^2 x_2 + \theta x_1^2 + u$$

and apply the control

$$u = -2x_1^2 x_2 + v,$$

which guarantees global asymptotic stability of the origin $(x_1, x_2) = (0, 0)$ with the discontinuous feedback $v = -(2x_1^2 + \rho_0)\text{sgn}(s)$.

If we apply the continuous approximation $v = -(2x_1^2 + \rho_0)\sigma_\varepsilon(s)$ we achieve convergence to a set which shrinks to the origin as $\varepsilon \to 0$.

Tracking Control

Consider a model represented in the normal form for input-output linearization:

$$\begin{aligned} \dot{z} &= f_0(z,\zeta) \\ \dot{\zeta}_1 &= \zeta_2 \\ &\vdots \\ \dot{\zeta}_{r-1} &= \zeta_r \\ \dot{\zeta}_r &= b(z,\zeta) + a(z,\zeta)u \\ y &= \zeta_1, \end{aligned}$$

where $a(z,\zeta)$ and $b(z,\zeta)$ are imperfectly known, but

$$a(z,\zeta) \geq g_0 > 0$$

with some positive constant g_0. In addition we assume the zero dynamics subsystem $\dot{z} = f_0(z,\zeta)$ is ISS with respect to ζ.

This system is of the general form (1) with $\eta = [z^T, \zeta_1, \cdots, \zeta_{r-1}]^T$ and $\omega = \zeta_r$, and we can design a virtual control

$$\zeta_r = -k_{r-1}\zeta_{r-1} - \cdots - k_1\zeta_1 \tag{3}$$

with coefficients $k_{r_1}, \cdots k_1$ such that

$$\begin{bmatrix} \dot{\zeta}_1 \\ \vdots \\ \vdots \\ \dot{\zeta}_{r-1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ & & \ddots & 1 \\ -k_1 & \cdots & \cdots & -k_{r-1} \end{bmatrix}}_{=:\, A_0} \begin{bmatrix} \zeta_1 \\ \vdots \\ \vdots \\ \zeta_{r-1} \end{bmatrix} \tag{4}$$

is asymptotically stable.

The dynamics restricted to the sliding surface (3) consist of the subsystem (4) driving the ISS zero dynamics; therefore the trajectories converge to the origin. Finite time convergence to the surface is achieved with the standard design approach discussed on page 1.

When the goal is to ensure that the output ζ_1 tracks the desired trajectory $y_d(t)$, we define the tracking error variables

$$e_1 := \zeta_1 - y_d(t),\ e_2 := \zeta_2 - \dot{y}_d(t),\ \cdots, e_r := \zeta_r - y_d^{(r-1)}(t)$$

and rewrite the system equations as

$$\begin{aligned}\dot{z} &= f_0(z, e_1 + y_d(t), \dots, e_r + y_d^{(r-1)}(t)) \\ \dot{e}_1 &= e_2 \\ &\vdots \\ \dot{e}_{r-1} &= e_r \\ \dot{e}_r &= b(z,\zeta) - y_d^{(r)}(t) + a(z,\zeta)u.\end{aligned}$$

As the sliding surface we select

$$s := e_r + k_{r-1}e_{r-1} + \cdots + k_1 e_1 = 0, \tag{5}$$

where $k_{r-1}, \cdots k_1$ are such that the matrix A_0 defined in (4) has all eigenvalues with negative real parts. Thus, $e(t) \to 0$ on the sliding surface and $z(t)$ remains bounded by the ISS assumption, and by the boundedness of $y_d(t)$ and its derivatives.

For the reaching phase we note that

$$\begin{aligned}\dot{s} &= \dot{e}_r + k_{r-1}\dot{e}_{r-1} + \cdots + k_1\dot{e}_1 \\ &= b(z,\zeta) - y_d^{(r)}(t) + k_{r-1}e_r + \cdots + k_1 e_2 + a(z,\zeta)u,\end{aligned}$$

and select

$$u = -\frac{1}{\hat{a}(z,\zeta)}\left[\hat{b}(z,\zeta) - y_d^{(r)}(t) + k_{r-1}e_r + \cdots + k_1 e_2\right] + v. \tag{6}$$

This yields

$$\dot{s} = \underbrace{\left(1 - \frac{a(z,\zeta)}{\hat{a}(z,\zeta)}\right)[\,\cdots\,] + (b(z,\zeta) - \hat{b}(z,\zeta))}_{=:\,\Delta(z,\zeta,t)} + a(z,\zeta)v$$

where $[\,\cdots\,]$ is the square bracketed term in (6), and $\Delta(z,\zeta,t)$ depends on t due to the derivatives of $y_d(t)$ occuring in this expression.

We then choose $\rho(z,\zeta,t)$ such that

$$\frac{|\Delta(z,\zeta,t)|}{a(z,\zeta)} \le \rho(z,\zeta,t)$$

and complete the design (6) with

$$v = -(\rho(z,\zeta,t) + \rho_0)\mathrm{sgn}(s), \;\; \rho_0 > 0. \tag{7}$$

Note that, if we set $y_d(t) \equiv 0$, the tracking controller (6)-(7) reduces to a stabilizing controller for the origin $(z,\zeta) = 0$.

Example: Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 + \sin x_1 \\ \dot{x}_2 &= \theta_1 x_1^2 + (1+\theta_2)u \quad |\theta_1| \le 2, \; |\theta_2| \le 0.5, \\ y &= x_1.\end{aligned} \tag{8}$$

To design a tracking controller we first bring the system to the normal form with the new variables $\zeta_1 = x_1$ and $\zeta_2 = x_2 + \sin x_1$:

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= (x_2 + \sin x_1)\cos x_1 + \theta_1 x_1^2 + (1+\theta_2)u. \end{aligned} \tag{9}$$

Then we define the error variables $e_1 = \zeta_1 - y_d(t)$ and $e_2 = \zeta_2 - \dot{y}_d(t)$, which are governed by

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= (x_2 + \sin x_1)\cos x_1 - \ddot{y}_d(t) + \theta_1 x_1^2 + (1+\theta_2)u, \end{aligned}$$

and select the sliding surface

$$s := e_2 + k_1 e_1 = 0, \quad k_1 > 0.$$

Thus,

$$\dot{s} = (x_2 + \sin x_1)\cos x_1 - \ddot{y}_d(t) + k_1 e_2 + \theta_1 x_1^2 + (1+\theta_2)u$$

and the feedback

$$\begin{aligned} u &= -(x_2 + \sin x_1)\cos x_1 + \ddot{y}_d(t) - k_1 e_2 + v \\ v &= -(\rho(x,t) + \rho_0) \quad \rho_0 > 0 \end{aligned}$$

results in

$$\dot{s} = \underbrace{\theta_2(-(x_2 + \sin x_1)\cos x_1 + \ddot{y}_d(t) - k_1 e_2) + \theta_1 x_1^2}_{=:\, \Delta(x_1, x_2, t)} + (1+\theta_2)v.$$

Using the bounds $|\theta_1| \leq 2$, $|\theta_2| \leq 0.5$ we get

$$\begin{aligned} \frac{|\Delta(x_1, x_2, t)|}{1+\theta_2} &\leq \frac{0.5|(x_2 + \sin x_1)\cos x_1 - \ddot{y}_d(t) + k_1 e_2| + 2x_1^2}{0.5} \\ &= |(x_2 + \sin x_1)\cos x_1 - \ddot{y}_d(t) + k_1 e_2| + 4x_1^2 \end{aligned}$$

and, substituting $e_2 = \zeta_2 - \dot{y}_d(t) = x_2 + \sin x_1 - \dot{y}_d(t)$, we select

$$\rho(x,t) = |(x_2 + \sin x_1)\cos x_1 - \ddot{y}_d(t) + k_1(x_2 + \sin x_1 - \dot{y}_d(t)| + 4x_1^2.$$

It is important to note that sliding mode control can address only limited forms of uncertainty. In the example (8) the uncertain terms appear in the same equation as the control input; that is, they are "matched" to the input. The first equation in (8) contains no uncertainty, which allowed us to bring the system to the normal form (9).