Center Manifold Theory

\[ \dot{x} = f(x) \quad f(0) = 0 \]  \hspace{1cm} (1)

Suppose \( A \triangleq \left. \frac{\partial f}{\partial x} \right|_{x=0} \) has \( k \) eigenvalues with zero real parts, and \( m = n - k \) eigenvalues with negative real parts.

Define \( \begin{bmatrix} y \\ z \end{bmatrix} = Tx \) such that

\[
TAT^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}
\]

where the eigenvalues of \( A_1 \) have zero real parts and the eigenvalues of \( A_2 \) have negative real parts.

Rewrite \( \dot{x} = f(x) \) in the new coordinates:

\[
\begin{align*}
\dot{y} &= A_1y + g_1(y,z) \\
\dot{z} &= A_2z + g_2(y,z)
\end{align*}
\]  \hspace{1cm} (2)

\( g_i(0,0) = 0, \frac{\partial g_i}{\partial y}(0,0) = 0, \frac{\partial g_i}{\partial z}(0,0) = 0, i = 1, 2. \)

**Theorem 1:** There exists an invariant manifold \( z = h(y) \) defined in a neighborhood of the origin such that

\[ h(0) = 0 \quad \frac{\partial h}{\partial y}(0) = 0. \]

**Reduced System:**

\[ \dot{y} = A_1y + g_1(y, h(y)) \quad y \in \mathbb{R}^k \]

**Theorem 2:** If \( y = 0 \) is asymptotically stable (resp., unstable) for the reduced system, then \( x = 0 \) is asymptotically stable (resp., unstable) for the full system \( \dot{x} = f(x) \).
Characterizing the Center Manifold

Define \( w \triangleq z - h(y) \) and note that it satisfies
\[
\dot{w} = A_2z + g_2(y, z) - \frac{\partial h}{\partial y}(A_1y + g_1(y, z)).
\]
The invariance of \( z = h(y) \) means that \( w = 0 \) implies \( \dot{w} = 0 \). Thus, the expression above must vanish when we substitute \( z = h(y) \):
\[
A_2h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y}(A_1y + g_1(y, h(y))) = 0.
\]
To find \( h(y) \) solve this differential equation for \( h \) as a function on \( y \).

If the exact solution is unavailable, an approximation is possible. For scalar \( y \), expand \( h(y) \) as
\[
h(y) = h_2y^2 + \cdots + h_py^p + O(y^{p+1})
\]
where \( h_1 = h_0 = 0 \) because \( h(0) = \frac{\partial h}{\partial y}(0) = 0 \). The notation \( O(y^{p+1}) \) refers to the higher order terms of power \( p + 1 \) and above.

Example:
\[
\begin{align*}
\dot{y} &= yz \\
\dot{z} &= -z + ay^2 & a \neq 0
\end{align*}
\]
This is of the form \((a)\) with \( g_1(y,z) = yz, g_2(y,z) = ay^2, A_2 = -1 \). Thus \( h(y) \) must satisfy
\[
-h(y) + ay^2 - \frac{\partial h}{\partial y}yh(y) = 0.
\]
Try \( h(y) = h_2y^2 + O(y^3) \):
\[
\begin{align*}
0 &= -h_2y^2 + O(y^3) + ay^2 - (2h_2y + O(y^2))y(h_2^2 + O(y^3)) \\
&= (a - h_2)y^2 + O(y^3) \\
\implies h_2 &= a
\end{align*}
\]
Reduced System: \( \dot{y} = y(ay^2 + O(y^3)) = ay^3 + O(y^4) \).
If \( a < 0 \), the full systems is asymptotically stable. If \( a > 0 \) unstable.
**Discrete-Time Models and a Chaos Example**

CT: \( \dot{x}(t) = f(x(t)) \)  
\( f(x^*) = 0 \)

DT: \( x_{n+1} = f(x_n) \) \( n = 0, 1, 2, \ldots \)  
\( f(x^*) = x^* \) ("fixed point")

Asymptotic stability criterion:  
\[ \Re \lambda_i(A) < 0 \] where  
\[ A \triangleq \left. \frac{\partial f}{\partial x} \right|_{x=x^*} \]
\[ f'(x^*) < 0 \] for first order system

These criteria are inconclusive if the respective inequality is not strict, but for first order systems we can determine stability graphically:

**Cobweb Diagrams for First Order Discrete-Time Systems**

Example: \( x_{n+1} = \sin(x_n) \) has unique fixed point at 0. Stability test above inconclusive since \( f'(0) = 1 \). However, the "cobweb" diagram below illustrates the convergence of iterations to 0:

In discrete time, even first order systems can exhibit oscillations:
Detecting Cycles Analytically

\[ f(p) = q \quad f(q) = p \quad \implies \quad f(f(p)) = p \quad f(f(q)) = q \]

For the existence of a period-2 cycle, the map \( f(f(\cdot)) \) must have two fixed points in addition to the fixed points of \( f(\cdot) \).

Period-3 cycles: fixed points of \( f(f(f(\cdot))) \).

Chaos in a Discrete Time Logistic Growth Model

\[ x_{n+1} = r(1 - x_n) x_n \]  \hspace{1cm} (3)

Range of interest: \( 0 \leq x \leq 1 \) \hspace{0.5cm} (\( x_n > 1 \implies x_{n+1} < 0 \))

We will study the range \( 0 \leq r \leq 4 \) so that \( f(x) = r(1 - x)x \) maps \([0, 1]\) onto itself.

Fixed points: \( x = r(1 - x)x \implies \begin{cases} x^* = 0 & \text{and} \\ x^* = 1 - \frac{1}{r} & \text{if } r > 1. \end{cases} \)

\( r \leq 1 \): \( x^* = 0 \) unique and stable fixed point

\( r > 1 \): \( x = 0 \) unstable because \( f'(0) = r > 1 \)
Note that a transcritical bifurcation occurred at \( r = 1 \), creating the new equilibrium

\[ x^* = 1 - \frac{1}{r} \]

Evaluate its stability using \( f'(x^*) = r(1 - 2x^*) = 2 - r \).

\[
\begin{align*}
  r < 3 & \Rightarrow |f'(x^*)| < 1 \text{ (stable)} \\
  r > 3 & \Rightarrow |f'(x^*)| > 1 \text{ (unstable)}.
\end{align*}
\]

At \( r = 3 \), a period-2 cycle is born:

\[
\begin{align*}
  x &= f(f(x)) \\
  &= r(1 - f(x))f(x) \\
  &= r(1 - r(1 - x))r(1 - x)x \\
  &= r^2x(1 - x)(1 - r + rx - rx^2) \\
  0 &= r^2x(1 - x)(1 - r + rx - rx^2) - x
\end{align*}
\]

Factor out \( x \) and \( (x - 1 + \frac{1}{r}) \), find the roots of the quotient:

\[
\begin{align*}
p, q &= \frac{r + 1 \pm \sqrt{(r - 3)(r + 1)}}{2r}
\end{align*}
\]

This period-2 cycle is stable when \( r < 1 + \sqrt{6} = 3.4494 \):

\[
\left. \frac{d}{dx} f(f(x)) \right|_{x=p} = f'(f(p))f'(p) = f'(p)f'(q) = 4 + 2r - r^2
\]

\[
|4 + 2r - r^2| < 1 \Rightarrow 3 < r < 1 + \sqrt{6} = 3.4494
\]

At \( r = 3.4494 \), a period-4 cycle is born!

"period doubling bifurcations"
\[ r_1 = 3 \quad \text{period-2 cycle born} \]
\[ r_2 = 3.4494 \quad \text{period-4 cycle born} \]
\[ r_3 = 3.544 \quad \text{period-8 cycle born} \]
\[ r_4 = 3.564 \quad \text{period-16 cycle born} \]

\[ \vdots \]
\[ r_\infty = 3.5699 \]

After \( r > r_\infty \), chaotic behavior for a window of \( r \), followed by windows of periodic behavior (e.g., period-3 cycle around \( r = 3.83 \)).

Below is the cobweb diagram for \( r = 3.9 \) which is in the chaotic regime: