Mathematical Background

\[ \dot{x} = f(x) \quad x(0) = x_0 \] (1)

Do solutions exist? Are they unique?

• If \( f(\cdot) \) is continuous (\( C^0 \)) then a solution exists, but \( C^0 \) is not sufficient for uniqueness.
  
  Example: \( \dot{x} = x^{1/3} \) with \( x(0) = 0 \)
  
  \[ x(t) \equiv 0, \quad x(t) = \left( \frac{2}{3} t \right)^{2/3} \] are both solutions

\[ x^{1/3} \]\hspace{1cm} \infty \text{ slope at } x = 0

• Sufficient condition for uniqueness: “Lipschitz continuity” (more restrictive than \( C^0 \))

\[ |f(x) - f(y)| \leq L|x - y| \] (2)

Definition: \( f(\cdot) \) is locally Lipschitz if every point \( x^0 \) has a neighborhood where (2) holds for all \( x, y \) in this neighborhood and for all \( t \) for some \( L \).

Example: \( (\cdot)^{1/3} \) is NOT locally Lipschitz (due to \( \infty \) slope)

\( (\cdot)^{1/3} \) is locally Lipschitz:

\[
x^3 - y^3 = (x^2 + xy + y^2) (x - y)
\]

in any nbhd of \( x^0 \), we can find \( L \) to upper bound this

\[ \Rightarrow |x^3 - y^3| \leq L|x - y| \]
• If \( f(\cdot) \) is continuously differentiable (\( C^1 \)), then it is locally Lipschitz.

Examples: \( x^3, x^2, e^x \), etc.

The converse is not true: local Lipschitz \( \not\Rightarrow C^1 \)

Example:

Not differentiable at \( x = \mp 1 \), but locally Lipschitz:

\[
|\text{sat}(x) - \text{sat}(y)| \leq |x - y| \quad (L = 1).
\]

Definition continued: \( f(\cdot) \) is globally Lipschitz if (2) holds \( \forall x, y \in \mathbb{R}^n \) (i.e., the same \( L \) works everywhere).

Examples: \( \text{sat}(\cdot) \) is globally Lipschitz. \( \cdot^3 \) is not globally Lipschitz:

• Suppose \( f(\cdot) \) is \( C^1 \). Then it is globally Lipschitz iff \( \frac{\partial f}{\partial x} \) is bounded.

\[
L = \sup_x |f'(x)|
\]
Preview of existence theorems:

1. \( f(\cdot) \) is \( C^0 \) \( \implies \) existence of solution \( x(t) \) on finite interval \([0, t_f)\).
2. \( f(\cdot) \) locally Lipschitz \( \implies \) existence and uniqueness on \([0, t_f)\).
3. \( f(\cdot) \) globally Lipschitz \( \implies \) existence and uniqueness on \([0, \infty)\).

Examples:
- \( \dot{x} = x^2 \) (locally Lipschitz) admits unique solution on \([0, t_f)\), but \( t_f < \infty \) from Lecture 1 (finite escape).
- \( \dot{x} = Ax \) globally Lipschitz, therefore no finite escape
  \[ |Ax - Ay| \leq L|x - y| \quad \text{with} \quad L = \|A\| \]

The rest of the lecture introduces concepts that are used in proving the existence theorems mentioned above.

Normed Linear Spaces

**Definition:** \( X \) is a normed linear space if there exists a real-valued norm \( |\cdot| \) satisfying:

1. \(|x| \geq 0 \quad \forall x \in X, \quad |x| = 0 \iff x = 0\).
2. \(|x + y| \leq |x| + |y| \quad \forall x, y \in X\) (triangle inequality)
3. \(|\alpha x| = |\alpha| \cdot |x| \quad \forall \alpha \in \mathbb{R} \) and \( x \in X\).

**Definition:** A sequence \( \{x_k\} \) in \( X \) is said to be a Cauchy sequence if

\[ |x_k - x_m| \to 0 \text{ as } k, m \to \infty. \quad (3) \]

Every convergent sequence is Cauchy. The converse is not true.

**Definition:** \( X \) is a Banach space if every Cauchy sequence converges to an element in \( X \).

All Euclidean spaces are Banach spaces.

**Example:**
\( C^n[a, b] \): the set of all continuous functions \([a, b] \to \mathbb{R}^n\) with norm:

\[ |x|_C = \max_{t \in [a,b]} |x(t)| \]

1. \(|x|_C \geq 0 \) and \(|x|_C = 0 \iff x(t) \equiv 0\).
2. \(|x + y|_C = \max_{t \in [a,b]} |x(t) + y(t)| \leq \max_{t \in [a,b]} \{ |x(t)| + |y(t)| \} \leq |x|_C + |y|_C
\)

3. \(|\alpha \cdot x|_C = \max_{t \in [a,b]} |\alpha| \cdot |x(t)| = |\alpha| \cdot |x|_C\)

It can be shown that \(C^n[a,b]\) is a Banach space.

**Fixed Point Theorems**

\[ T(x) = x \]  \hspace{1cm} (4)

**Brouwer’s Theorem** (Euclidean spaces):

If \(U\) is a closed bounded subset of a Euclidean space and \(T : U \to U\) is continuous, then \(T\) has a fixed point in \(U\).

**Schauder’s Theorem** (Brouwer’s Thm \(\to\) Banach spaces):

If \(U\) is a closed bounded convex subset of a Banach space \(X\) and \(T : U \to U\) is completely continuous\(^2\), then \(T\) has a fixed point in \(U\).

**Contraction Mapping Theorem:**

If \(U\) is a closed subset of a Banach space and \(T : U \to U\) is such that

\[ |T(x) - T(y)| \leq \rho |x - y| \quad \rho < 1 \quad \forall x, y \in U \]

then \(T\) has a unique fixed point in \(U\) and the solutions of \(x_{n+1} = T(x_n)\) converge to this fixed point from any \(x_0 \in U\).

**Example:** The logistic map (Lecture 5)

\[ T(x) = rx(1 - x) \]  \hspace{1cm} (5)

with \(0 \leq r \leq 4\) maps \(U = [0, 1]\) to \(U\). \(|T'(x)| \leq r \quad \forall x \in [0, 1]\), so the contraction property holds with \(\rho = r\).

If \(r < 1\), the contraction mapping theorem predicts a unique fixed point that attracts all solutions starting in \([0, 1]\).
Proof steps for the Contraction Mapping Thm:

1. Show that \( \{x_n\} \) formed by \( x_{n+1} = T(x_n) \) is a Cauchy sequence. Since we are in a Banach space, this implies a limit \( x^* \) exists.

2. Show that \( x^* = T(x^*) \).

3. Show that \( x^* \) is unique.

Details of each step:

1. \[
|x_{n+1} - x_n| = |T(x_n) - T(x_{n-1})| \leq \rho |x_n - x_{n-1}|
\leq \rho^2 |x_{n-1} - x_{n-2}|
\leq \cdots
\leq \rho^n |x_1 - x_0|.
\]

Since \( \frac{\rho^n}{1 - \rho} \to 0 \) as \( n \to \infty \), we have \( |x_{n+r} - x_n| \to 0 \) as \( n \to \infty \).

2. \[
|x^* - T(x^*)| = |x^* - x_n + T(x_{n-1}) - T(x^*)| \leq |x^* - x_n| + |T(x_{n-1}) - T(x^*)| \leq |x^* - x_n| + \rho |x^* - x_{n-1}|.
\]

Since \( \{x_n\} \) converges to \( x^* \), we can make this upper bound arbitrarily small by choosing \( n \) sufficiently large. This means that \( |x^* - T(x^*)| = 0 \), hence \( x^* = T(x^*) \).

3. Suppose \( y^* = T(y^*) \) \( y^* \neq x^* \).

\[
|x^* - y^*| = |T(x^*) - T(y^*)| \leq \rho |x^* - y^*| \implies x^* = y^*.
\]

Thus we have a contradiction.