

EE C222/ME C237 - Spring'18 - Lecture 9 Notes¹

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LaSalle-Krasovskii Invariance Principle

- Applicable to time-invariant systems.
- Allows us to conclude asymptotic stability from $\dot{V}(x) \leq 0$ if additional conditions hold:

Suppose $\Omega_c = \{x : V(x) \leq c\}$ is bounded and $\dot{V}(x) \leq 0$ in Ω_c . Define $S = \{x \in \Omega_c : \dot{V}(x) = 0\}$ and let M be the largest invariant set in S . Then, for every $x(0) \in \Omega_c$, $x(t) \rightarrow M$.

Corollary: If no solution other than $x(t) \equiv 0$ can stay identically in S then $M = \{0\}$ and we conclude asymptotic stability.

Example (from last lecture):

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - g(x_1) \quad a > 0, xg(x) > 0 \quad \forall x \neq 0 \end{aligned} \tag{1}$$
$$V(x) = \int_0^{x_1} g(y)dy + \frac{1}{2}x_2^2 \implies \dot{V}(x) = -ax_2^2$$
$$S = \{x \in \Omega_c | x_2 = 0\}$$

If $x(t)$ stays identically in S , then $x_2(t) \equiv 0 \implies \dot{x}_2(t) \equiv 0 \implies g(x_1(t)) \equiv 0 \implies x_1(t) \equiv 0 \implies$ asymptotic stability from Corollary.

Example (linear system): Same system above with $g(x_1) = bx_1$:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - bx_1 \quad a > 0, b > 0 \end{aligned} \tag{2}$$

$V(x) = \frac{b}{2}x_1^2 + \frac{1}{2}x_2^2 \implies \dot{V}(x) = -ax_2^2 \implies$ Invariance Principle works as in the example above.

Alternatively, construct another Lyapunov function with negative definite $\dot{V}(x)$. Try $V(x) = x^T P x$ where $P = P^T > 0$ is to be selected.

$$\dot{V}(x) = x^T P \dot{x} + \dot{P} x = x^T (A^T P + P A) x \text{ where } A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}$$

Let $P = \frac{1}{2} \begin{bmatrix} b & \epsilon \\ \epsilon & 1 \end{bmatrix}$, that is $V(x) = \frac{b}{2}x_1^2 + \epsilon x_1 x_2 + \frac{1}{2}x_2^2$.

Note that $P > 0$ if $\epsilon^2 < b$.

$$A^T P + PA = \begin{bmatrix} -\epsilon b & -\epsilon a/2 \\ -\epsilon a/2 & \epsilon - a \end{bmatrix} \begin{cases} \leq 0 \text{ if } \epsilon = 0 \\ < 0 \text{ if } 0 < \epsilon < a \text{ and } \underbrace{\epsilon b(a - \epsilon)}_{0 < \epsilon < \frac{ba}{b + \frac{a^2}{4}}} > \frac{\epsilon^2 a^2}{4} \end{cases}$$

Linear Systems

$$\dot{x} = Ax \quad x \in \mathbb{R}^n \tag{3}$$

Sastry (Sec. 5.7-5.8), Khalil (Sec. 4.3)

$x = 0$ is stable if $\Re\{\lambda_i(A)\} \leq 0$ for all $i = 1, \dots, n$ and eigenvalues on the imaginary axis have Jordan blocks of order one.² It is asymptotically stable if $\Re\{\lambda_i(A)\} < 0$ for all i , i.e., A is "Hurwitz."

² i.e., if λ is an eigenvalue of multiplicity q then $\lambda I - A$ must have rank $n - q$.

Example:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ is unstable: } \left. \begin{matrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = 0 \end{matrix} \right\} x_1(t) = x_1(0) + x_2(0)t$$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ is stable.}$$

Lyapunov Functions for Linear Systems

$$\begin{aligned} V(x) &= x^T P x & P &= P^T > 0 \\ \dot{V}(x) &= x^T (A^T P + PA)x \end{aligned} \tag{4}$$

If $\exists P = P^T > 0$ such that $A^T P + PA = -Q < 0$, then A is Hurwitz. The converse is also true:

Theorem: A is Hurwitz if and only if for any $Q = Q^T > 0$, there exists $P = P^T > 0$ such that

$$A^T P + PA = -Q. \tag{5}$$

Moreover, the solution P is unique.

(5) is known as the Lyapunov Equation. The Matlab command `lyap(A',Q)` returns the solution P .

Proof:

(if) From (4) above, the Lyapunov function $V(x) = x^T P x$ proves asymptotic stability which means A is Hurwitz.

(only if) Assume $\Re\{\lambda_i(A)\} < 0 \forall i$. Show $\exists P = P^T > 0$ such that $A^T P + PA = -Q$.

Candidate:

$$P = \int_0^\infty e^{A^T t} Q e^{A t} dt. \tag{6}$$

- The integral exists because $\|e^{At}\| \leq \kappa e^{-\alpha t}$.
- $P = P^T$
- $P > 0$ because $x^T P x = \int_0^\infty (e^{At} x)^T \underbrace{Q(e^{At} x)}_{\triangleq \phi(t,x)} dt \geq 0$ and
 $x^T P x = 0 \implies \phi(t, x) \equiv 0 \implies x = 0$ because e^{At} is nonsingular.
- $A^T P + P A = \int_0^\infty \underbrace{\left(A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A \right)}_{= \frac{d}{dt} (e^{At} Q e^{At})} dt$
 $= e^{A^T t} Q e^{At} \Big|_0^\infty = 0 - Q = -Q$

Uniqueness:

Suppose there is another $\hat{P} = \hat{P}^T > 0$ satisfying $\hat{P} \neq P$, and $A^T \hat{P} + \hat{P} A = -Q$.

$$\implies (P - \hat{P})A + A^T(P - \hat{P}) = 0$$

Define $W(x) = x^T(P - \hat{P})x$.

$$\frac{d}{dt} W(x(t)) = 0 \implies W(x(t)) = W(x(0)) \quad \forall t.$$

Since A is Hurwitz, $x(t) \rightarrow 0$ and $W(x(t)) \rightarrow 0$.

Combining the two statements above, we conclude $W(x(0)) = 0$ for any $x(0)$. This is possible only if $P - \hat{P} = 0$ which contradicts $\hat{P} \neq P$.

Invariance Principle Applied to Linear Systems

$$A^T P + P A = -Q \leq 0 \tag{7}$$

Can we conclude that A is Hurwitz if Q is only semidefinite?

Decompose Q as $Q = C^T C$ where $C \in \mathbb{R}^{r \times n}$, r is the rank of Q .

$$\dot{V}(x) = -x^T Q x = -x^T C^T C x = -y^T y$$

where $y \triangleq Cx$. The invariance principle guarantees asymptotic stability if

$$y(t) = Cx(t) \equiv 0 \implies x(t) \equiv 0.$$

This implications is true if the pair (C, A) is observable.

Example (beginning of the lecture):

$$A = \begin{bmatrix} 0 & 1 \\ -b & a \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \implies C = [0 \quad \sqrt{a}]$$

(C, A) is observable if $b \neq 0$.