EE 225A Spring 2011
HW 4 Solution

Problem 1:

We will show this by induction

Base case: 1x1 matrix
By construction, a scalar is an eigenvalue for the expression $Av = \lambda v$ with $A = \lambda, v = 1$

Inductive hypothesis: assume $A_n \in \mathbb{C}^{n\times n}$ is lower triangular (without loss of generality) and has eigenvalues $\{\lambda_i\}_{i=1}^{n}$ equal to the diagonal elements of A_n

Inductive step: Form $A_{n+1} \in \mathbb{C}^{(n+1)\times(n+1)}$ such that it is also lower triangular as follows

$$A_{n+1} = \begin{bmatrix} & & & 0 \\ & A_n & & \vdots \\ & & & 0 \\ a_{n+1,1} & \cdots & a_{n+1,n} & a_{n+1,n+1} \end{bmatrix}$$

Examine the characteristic polynomial expression

$$\det(A_{n+1} - \lambda_{n+1} I_{n+1}) = 0$$

Write down the cofactor expansion of the determinant

$$\det(A_{n+1} - \lambda_{n+1} I_{n+1}) = \left(a_{n+1,n+1} - \lambda_{n+1}\right) \det(A_n - \lambda_{n+1} I_n) + \sum_{i=1}^{n} a_{n+1,i} C_{n+1,i}$$

The second term, $\sum_{i=1}^{n} a_{n+1,i} C_{n+1,i} = 0$, because $\forall\ i = 1, \dots, n$, associated minor has a column of zeros. The $\det(A_n - \lambda_{n+1} I_n) \neq 0$ for distinct eigenvalues (otherwise more general case is handled by Jordan form) so we have

$$a_{n+1,n+1} - \lambda_{n+1} = 0 \Rightarrow \lambda_{n+1} = a_{n+1,n+1},$$

which is what we wanted to show, that the eigenvalues are the diagonal elements of the triangular matrix.

Problem 2

First, to show the relationship between the trace and the eigenvalues, we take advantage of the fact that R is diagonalizable

$$A = P\Lambda P^{-1}$$

$$tr(A) = tr(P\Lambda P^{-1}) = tr(P^{-1}P\Lambda) = tr(\Lambda) = \sum_{i=1}^{M} \lambda_i$$

We also used the property $tr(AB) = tr(BA)$
The correlation matrix for this problem has the following structure

$$R \in \begin{bmatrix} \sigma^2 & & r_{ji}^* \\ & \ddots & \\ r_{ij} & & \sigma^2 \end{bmatrix}$$

Therefore,

$$tr(R) = M\sigma^2 = \sum_{i=1}^{M} \lambda_i$$

Problem 3

3a)

Using the Karhunen-Loève expansion

$$c_i(n) = v_i^H u(n) = \frac{1}{\sqrt{M}} \exp\left(-j\frac{2\pi i}{M} n\right) u(n); \quad i = 0,1,\dots,M-1$$

3b)

The coefficients are uncorrelated. We check this by looking at the expectation

$$E\left[c_i(n)c_j^H(n)\right] = E\left[v_i^H u(n)u(n)^H v_j\right] = v_i^H E[u(n)u(n)^H]v_j = v_i^H R v_j$$

The parameter c(i)'s are not generally orthogonal . The only case that they are orthogonal is when the correlation matrix R can be diagonalized as follows:

$$E\left[c_i(n)c_j^H(n)\right] = v_i^H V \Lambda V^H v_j$$

Or in other words, the Fourier vectors are eigenvectors of the correlation matrix. In that special case we have:

$$E\left[c_i(n)c_j^H(n)\right] = \begin{cases} \lambda_i \; if \; i = j \\ 0 \; if \; i \neq j \end{cases}$$

3c)

The expectation $E[|c_i(n)|^2]$ approximates the eigenvalue associated with the Fourier coefficient c_i

3d)

$$e(n) = u(n) - \hat{u}(n) = \sum_{i=0}^{M-1} c_i(n)v_i - \sum_{i=0}^{P} c_i(n)v_i = \sum_{i=P+1}^{M-1} c_i(n)v_i$$

$$E[|e(n)|^2] = E\left[\sum_{i=P+1}^{M-1}\sum_{j=P+1}^{M-1} c_i(n)v_i v_j^H c_j^H(n)\right]$$

The only terms that are non-zero in the sum are when $i = j$ because of the orthonormal basis vectors v_i. Pulling the expectation into the sum,

$$E[|e(n)|^2] = E\left[\sum_{i=P+1}^{M-1} c_i(n)c_i^H(n)\right] = \sum_{i=P+1}^{M-1} E[c_i(n)c_i^H(n)] = \sum_{i=P+1}^{M-1} \lambda_i$$

Problem 4

4a)
Select $g(\cdot) = \ln(\cdot)$ and substitute into the Szego theorem

$$\lim_{M\to\infty} \frac{\ln(\lambda_1) + \cdots + \ln(\lambda_M)}{M} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\big(S(\omega)\big)\, d\omega$$

Now rewrite the LHS into a single natural log expression

$$\lim_{M\to\infty} \frac{1}{\mathrm{M}} \ln\left(\prod_{i}^{M} \lambda_i\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\big(S(\omega)\big)\, d\omega$$

Now note that the product of the eigenvalues is the determinant of R

$$\lim_{M\to\infty} \frac{1}{\mathrm{M}} \ln(\det R) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\big(S(\omega)\big)\, d\omega$$

Exponentiate both sides

$$\lim_{M\to\infty} [\det R]^{\frac{1}{M}} = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\big(S(\omega)\big)\, d\omega\right]$$

b)
Select $g(\cdot) = 1(\cdot)$, or the identity function map

$$\lim_{M\to\infty} \frac{\lambda_1 + \cdots + \lambda_M}{M} = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega) d\omega$$

Now notice that the sum of the eigenvalues is the trace

$$\lim_{M\to\infty} \frac{tr\ R}{M} = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega) d\omega$$

c) This quantity represents an estimate of the average variance of the random process.

Problem 5

5a)

$$P_0 = r(0) = E\left[\big(u(n) * w(n)\big)\big(u(n) * w(n)\big)^H\right]$$

$$= E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} w(n-i)u(n)u^H(n)w^H(n-j)\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} w^H(i)E[u(n)u^H(n)]w(j)$$

$$= w^H R w$$

Where we used the linearity of expectation and reversed time on the filter coefficients

b)
The same results as the above step apply to this part. Therefore, the autocorrelation of the noise is only a scalar, $R = \sigma^2$, so

$$N = w^H \sigma^2 w = \sigma^2$$

Using the fact that $w^H w = 1$.
c)

$$SNR = \frac{w^H R w}{\sigma^2}$$

Using the results of the Minimax theorem for eigenvalues

$$\max SNR = \sup_{\substack{w \in \mathbb{C}^n \\ \|w\|=1}} \frac{w^H R w}{\sigma^2} = \frac{\lambda_{max}}{\sigma^2} \; where \; w = q_{max}$$

d)

Using the results of the Minimax theorem for eigenvalues

$$\min SNR = \inf_{\substack{w \in \mathbb{C}^n \\ \|w\|=1}} \frac{w^H R w}{\sigma^2} = \frac{\lambda_{min}}{\sigma^2} \; where \; w = q_{min}$$