

- 4.11. We wish to design a 2-D FIR filter from a 1-D FIR filter by the transformation method. Suppose the transformation sequence  $t(n_1, n_2)$  is zero except for the five points shown in the following figure.

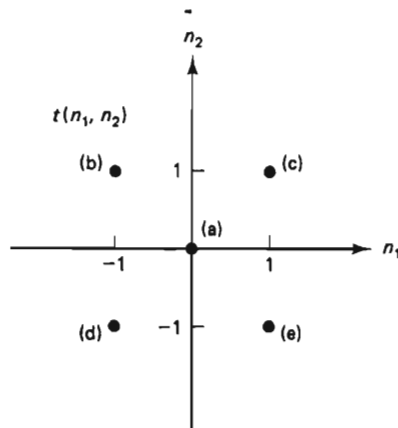


Figure P4.11

- (a) What are the minimum constraints on the coefficients  $a$ ,  $b$ ,  $c$ ,  $d$ , and  $e$  for the resulting 2-D filter to be zero phase [i.e., for the filter frequency response  $H(\omega_1, \omega_2)$  to be real]? Assume the 1-D filter used in the transformation method is zero phase.
- (b) Suppose  $a = b = e = 0$  and  $c = d = 1/2$ . Suppose also that the 1-D zero-phase filter used is a lowpass filter with a passband region given by  $-\pi/2 \leq \omega \leq \pi/2$ . Sketch the complete passband region (or regions) of the resulting 2-D filter. Label the axes in your sketch.
- 4.12. Suppose we design a 2-D FIR filter  $H(\omega_1, \omega_2)$  from a 1-D FIR filter  $H(\omega)$  by

$$H(\omega_1, \omega_2) = H(\omega)|_{\cos\omega = T(\omega_1, \omega_2) = \cos(\omega_1 - \omega_2)}.$$

- (a) Determine  $t(n_1, n_2)$ , the transformation sequence.
- (b) If  $H(\omega)$  is the frequency response of a highpass filter, what type of filter is  $H(\omega_1, \omega_2)$ ?
- 4.13. In FIR filter design by the frequency transformation method, the 2-D filter frequency response  $H(\omega_1, \omega_2)$  is obtained from the 1-D filter frequency response  $H(\omega)$  by

$$H(\omega_1, \omega_2) = H(\omega)|_{\cos\omega = T(\omega_1, \omega_2)}.$$

The filter designed by using the above transformation is typically not circularly symmetric. One clever student suggested the following approach:

$$H(\omega_1, \omega_2) = H(\omega)|_{\omega = \sqrt{\omega_1^2 + \omega_2^2}}.$$

Is this a good approach? If so, explain why. If not, discuss the problems of this approach.

- 4.14. Consider a  $3 \times 3$ -point transformation sequence  $t(n_1, n_2)$ . We design a 2-D zero-phase filter  $H(\omega_1, \omega_2)$  from a 1-D zero-phase filter  $H(\omega)$  by

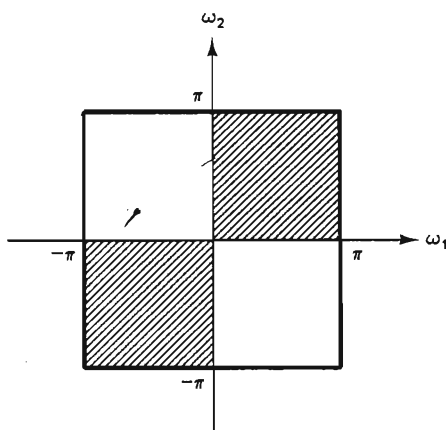
$$H(\omega_1, \omega_2) = H(\omega)|_{\cos \omega = \tau(\omega_1, \omega_2)}$$

where  $T(\omega_1, \omega_2)$  is the Fourier transform of  $t(n_1, n_2)$ .

- (a) If we impose the zero-phase constraint of  $t(n_1, n_2)$  such that  $t(n_1, n_2) = t(-n_1, -n_2)$ , show that  $T(\omega_1, \omega_2)$  can be expressed in the form

$$T(\omega_1, \omega_2) = A + B \cos \omega_1 + C \cos \omega_2 + D \cos \omega_1 \cos \omega_2 + E \sin \omega_1 \sin \omega_2.$$

- (b) We wish to design a filter whose ideal-frequency response  $H_d(\omega_1, \omega_2)$  is shown in the following figure.



$$H_d(\omega_1, \omega_2) = \begin{cases} 1, & \text{in the shaded region} \\ 0, & \text{in the unshaded region} \end{cases}$$

Figure P4.14

Determine one set of  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$  that could be used in the design of a 2-D filter.

- (c) With the choice of parameters in (b), what type of 1-D filter  $H(\omega)$  do we need to design the filter in (b)?
- 4.15. Suppose we design a 2-D zero-phase lowpass filter  $H(\omega_1, \omega_2)$  from a 1-D zero-phase filter  $H(\omega)$  by

$$H(\omega_1, \omega_2) = H(\omega)|_{\cos \omega = -1/2 + (\cos \omega_1)/2 + (\cos \omega_2)/2 + (\cos \omega_1 \cos \omega_2)/2}.$$

The 2-D filter  $H(\omega_1, \omega_2)$  has to satisfy the following specification:

$$0.95 \leq H(\omega_1, \omega_2) \leq 1.05, \quad (\omega_1, \omega_2) \in \text{passband region (see figure)}$$

$$-0.02 \leq H(\omega_1, \omega_2) \leq 0.02, \quad (\omega_1, \omega_2) \in \text{stopband region (see figure)}$$

Determine the filter specification that the 1-D filter  $H(\omega)$  has to satisfy so that the resulting 2-D filter  $H(\omega_1, \omega_2)$  will be guaranteed to meet the above filter specification.

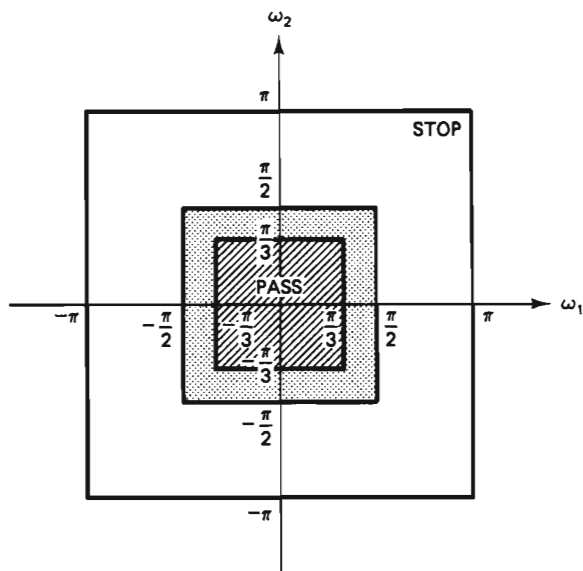


Figure P4.15

4.16. Consider a 1-D zero-phase FIR filter  $h(n)$  with length  $2N + 1$ . The frequency response  $H(\omega)$  can be expressed as

$$H(\omega) = \sum_{n=-N}^N h(n)e^{-j\omega n} \quad (1)$$

$$= \sum_{n=0}^N a(n) \cos \omega n \quad (2)$$

where  $a(0) = h(0)$  and  $a(n) = 2h(n)$  for  $n \geq 1$ . In this problem, we show that  $H(\omega)$  can also be expressed in the form of

$$H(\omega) = \sum_{n=0}^N b(n) (\cos \omega)^n \quad (3)$$

where  $b(n)$  can be simply related to  $a(n)$  or  $h(n)$ . Note that

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\text{and } \sin(A + B) = \sin A \cos B + \cos A \sin B.$$

- (a) Show that  $\cos 2\omega = 2 \cos \omega \cos \omega - 1$ .  
 (b) Show that  $\cos 3\omega = 2 \cos \omega \cos 2\omega - \cos \omega$ .  
 (c) More generally, show that for  $n \geq 2$ ,

$$\cos \omega n = 2 \cos \omega \cos \omega(n-1) - \cos \omega(n-2).$$

- (d) Let  $\cos \omega n$  be denoted by  $P_n[\cos \omega]$ . From the results in (a), (b), and (c), show that

$$\begin{aligned} P_0[\cos \omega] &= 1 \\ P_1[\cos \omega] &= \cos \omega \\ P_2[\cos \omega] &= 2 \cos \omega P_1[\cos \omega] - P_0[\cos \omega] \\ &\vdots \\ P_n[\cos \omega] &= 2 \cos \omega P_{n-1}[\cos \omega] - P_{n-2}[\cos \omega]. \end{aligned} \quad (4)$$

The polynomial  $P_n[\cdot]$  is known as the *Chebyshev polynomial*.

- (e) From the result of (d), show that  $\cos \omega n$  can be expressed as a linear combination of  $1, \cos \omega, (\cos \omega)^2, \dots, (\cos \omega)^n$ . In other words, show that

$$\cos \omega n = \sum_{k=0}^n q(k) (\cos \omega)^k \quad \text{for some } q(k). \quad (5)$$

- (f) Using (4), determine  $q(k)$  in (5) for  $\cos 3\omega$ .  
 (g) From (2) and (5), show that  $H(\omega)$  can be expressed in the form of (3).  
 (h) Suppose  $h(n)$  is given by

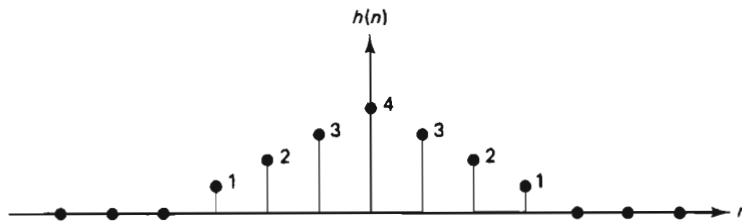


Figure P4.16

Determine  $a(n)$  in (2) and  $b(n)$  in (3).

- 4.17. We wish to design a 2-D zero-phase low-pass filter with design specification parameters  $\delta_p$  (passband tolerance),  $\delta_s$  (stopband tolerance),  $R_p$  (passband region), and  $R_s$  (stopband region). We assume that the impulse response of the filter designed has a region of support of  $(2N + 1) \times (2N + 1)$  points. Suppose we have developed an algorithm that solves the following problem:

Given  $R_p, R_s, k = \delta_p/\delta_s$ , and  $N$ , determine  $h(n_1, n_2)$  so that  $\delta_s$  is minimized.

We'll refer to the algorithm as Algorithm A.

- (a) Using Algorithm A, develop a method that solves the following problem:

Given  $R_p, R_s, \delta_p$ , and  $\delta_s$ , determine  $h(n_1, n_2)$  so that  $N$  is minimized.

(b) Suppose  $R_p$  is given by

$$\sqrt{\omega_1^2 + \omega_2^2} \leq \omega_p$$

and  $R_s$  is given by

$$\sqrt{\omega_1^2 + \omega_2^2} \geq \omega_s.$$

Using Algorithm A, develop a method that solves the following problem:

Given  $\omega_p$ ,  $\delta_p$ ,  $\delta_s$ , and  $N$ , determine  $h(n_1, n_2)$  so that  $\omega_s$  is minimized.

- 4.18.** Consider an FIR filter with a region of support of  $(2N + 1) \times (2N + 1)$  points. If we implement the filter by direct convolution, the number of multiplications required per output point will be  $(2N + 1)^2$ . If  $h(n_1, n_2)$  has some symmetry, this symmetry can be exploited to reduce the number of multiplications. Suppose  $h(n_1, n_2)$  has a fourfold symmetry given by

$$h(n_1, n_2) = h(-n_1, n_2) = h(n_1, -n_2).$$

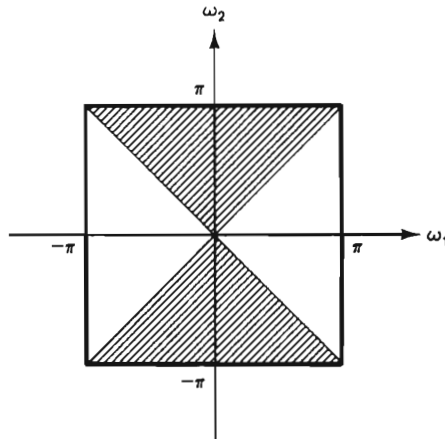
Discuss how this can be used to reduce the number of multiplications. Approximately how many multiplications can be reduced in this case? Is it possible to reduce the number of additions required by exploiting the symmetry?

- 4.19.** Suppose we define an  $(N + 1) \times 1$ -column vector  $\mathbf{V}(\boldsymbol{\omega}) = \mathbf{V}(\omega_1, \omega_2)$  at a particular frequency  $\boldsymbol{\omega}$  by

$$\mathbf{V}(\boldsymbol{\omega}) = [\phi_0(\boldsymbol{\omega}), \phi_1(\boldsymbol{\omega}), \phi_2(\boldsymbol{\omega}), \dots, \phi_N(\boldsymbol{\omega})]^T.$$

If  $\mathbf{V}(\boldsymbol{\omega}_i)$  for  $0 \leq i \leq N$  are independent vectors for any choice of distinctly different  $\boldsymbol{\omega}_i \in K$  where  $K$  is some known region, then the functions  $\phi_i(\boldsymbol{\omega})$  are said to form a Chebyshev set or satisfy the Haar condition. Show that  $\phi_i(\boldsymbol{\omega}) = \cos(\omega_1 n_1^i + \omega_2 n_2^i)$  for integers  $n_1^i$  and  $n_2^i$  do not satisfy the Haar condition for  $K$  given by  $0 \leq \omega_1 \leq \pi$ ,  $0 \leq \omega_2 \leq \pi$ .

- 4.20.** In some geophysical applications, it is useful to design a *fan filter* whose idea frequency response  $H_d(\omega_1, \omega_2)$  is shown in the following figure.



$$H_d(\omega_1, \omega_2) = \begin{cases} 1, & \text{in the shaded region} \\ 0, & \text{in the unshaded region} \end{cases}$$

Figure P4.20

One approach suggested is to design a 2-D zero-phase fan filter  $H(\omega_1, \omega_2)$  from a 1-D zero-phase filter  $H(\omega)$  by

$$H(\omega_1, \omega_2) = H(\omega)|_{\cos \omega = T(\omega_1, \omega_2) = (\cos \omega_1)/2 - (\cos \omega_2)/2}.$$

- (a) What type of 1-D filter do we need?  
 (b) Suppose the boundaries of the passband and stopband regions in the fan filter specification coincide with the contours given by  $\cos \omega = T(\omega_1, \omega_2)$ . Also suppose that the 1-D filter is designed by using the Parks-McClellan algorithm. Is the 2-D filter  $H(\omega_1, \omega_2)$  designed optimal in the Chebyshev sense?
- 4.21. Suppose an FIR filter that meets its design specification has been obtained. The region of support of the digital filter  $h(n_1, n_2)$  is  $25 \times 25$  points in size. We consider realizing this filter using the overlap-add method to exploit the computational efficiency of an FFT algorithm. The size of the DFT used is  $N \times N$ , where  $N$  can be expressed as a power of 2. What is the approximate number of multiplications per output point in terms of  $N$ ? Assume that the region of support of the input  $x(n_1, n_2)$  is very large and that the row-column decomposition method with a decimation-in-time FFT algorithm for the 1-D DFT computation is used.
- 4.22. The frequency response  $H(\omega_1, \omega_2)$  of an FIR filter designed by the transformation method can be expressed as

$$H(\omega_1, \omega_2) = \sum_{n=0}^N a(n) \cos \omega n|_{\cos \omega = T(\omega_1, \omega_2)} \quad (1)$$

or

$$H(\omega_1, \omega_2) = \sum_{n=0}^N b(n) (\cos \omega)^n|_{\cos \omega = T(\omega_1, \omega_2)}. \quad (2)$$

In Section 4.6.2, we developed an implementation method based directly on (2). An alternate implementation method can be developed by using (1) and expressing  $\cos \omega n$  in terms of  $\cos \omega(n-1)$  and  $\cos \omega(n-2)$ . Specifically, let  $\cos \omega n$  be denoted by  $P_n[\cos \omega]$ . From (1),

$$H(\omega_1, \omega_2) = \sum_{n=0}^N a(n) P_n[T(\omega_1, \omega_2)]. \quad (3)$$

From Problem 4.16,  $P_n[\bullet]$  is the  $n$ th-order Chebyshev polynomial and is given by

$$\begin{aligned} P_0[x] &= 1 \\ P_1[x] &= x \\ P_2[x] &= 2xP_1[x] - P_0[x] \\ &\vdots \\ P_n[x] &= 2xP_{n-1}[x] - P_{n-2}[x]. \end{aligned} \quad (4)$$

- (a) Show that  $P_n[T(\omega_1, \omega_2)]$  for  $n \geq 2$  can be obtained from  $P_{n-1}[T(\omega_1, \omega_2)]$  and  $P_{n-2}[T(\omega_1, \omega_2)]$  by