

Multi-Resolution Expansion

- Scaling f_n Φ : creates a series of approximations of a f_n each differing by a factor of 2 in resolution

- Function φ : (wavelet) encodes diff between adjacent approximations.

Series Expansion

Expand f_n $f(x)$ as:

$$f(x) = \sum_k \alpha_k \varphi_k(x)$$

$\varphi_k(x)$ \triangleq real valued expansion functions
 α_k \triangleq " " " coefficients

If expansion unique i.e. only one set of α_k for $f(x)$

$\Rightarrow \phi_k =$ basis function.

$\{\phi_k\} =$ basis for class of fns.

function space: $V \triangleq \text{Span}_k \{ \phi_k(x) \}$

closed span of expansion set

$f(x) \in V \Rightarrow f(x)$ is in closed span of $\{\phi_k(x)\}$

and can be written as $f(x) = \sum_k \alpha_k \phi_k(x)$

Dual function $\{ \underbrace{\phi_k(x)}_n \}$ To $\{ \phi_k(x) \}$

$$\alpha_k = \langle \phi_k(x), f(x) \rangle = \int \phi_k^*(x) f(x) dx$$

Consider 3 cases:

① Expansion f_n s form an orthonormal basis for

V :

$$\langle \phi_j(x), \phi_k(x) \rangle = \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

$\Rightarrow \phi_k(x) = \check{\phi}_k(x)$ basis & dual same.

$$\Rightarrow \alpha_k = \langle \phi_k(x), f(x) \rangle$$

② Expansion f_n orthogonal but not orthonormal

$$\langle \phi_j(x), \phi_k(x) \rangle = 0 \quad j \neq k$$

\Rightarrow basis f_n and dual are bi-orthogonal

$$\alpha_k = \langle \check{\phi}_k(x), f(x) \rangle$$

$$\langle \phi_j(x), \check{\phi}_k(x) \rangle = \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

③ More than one set of α_k in

$$f(x) = \sum_k \alpha_k \phi_k(x)$$

\Rightarrow Exp f_n and dual are "over complete"

or "redundant"

Form a frame

$$A \|f(x)\|^2 \leq \sum_k |\langle \phi_k(x), f(x) \rangle|^2 \leq B \|f(x)\|^2$$

for $A > 0, B < \infty \quad \forall f(x) \in V$

- If $A=B \rightarrow$ Tight frame.

Parseval's 1992 $f(x) = \frac{1}{A} \sum_k \langle \phi_k(x), f(x) \rangle \phi_k(x)$

Scaling functions

- Start with real, square integrable $f \in \mathcal{L}^2(\mathbb{R})$

- Build a set $\phi_{j,k}(x) = 2^{j/2} \phi(2^j x - k)$

$j, k \in \mathbb{Z}$ $\phi(x) \in \mathcal{L}^2(\mathbb{R})$

Denote subspace ~~spaces~~:

$$V_j = \text{span} \{ \phi_{j,k}(x) \}$$

Then if $f(x) \in V_j \Rightarrow f(x) = \sum_k \alpha_k \phi_{j,k}(x)$

Haar basis.

Example

$$0 < x < 1$$

$$\phi(x) = \begin{cases} 1 \\ 0 \end{cases} \text{ otherwise}$$

Show Fig 7.11 $\phi + \omega \in E$

~~with~~ $f(x) \in V_0 \Rightarrow f(x) \in V_1 : V_0 \subset V_1$

For Haar

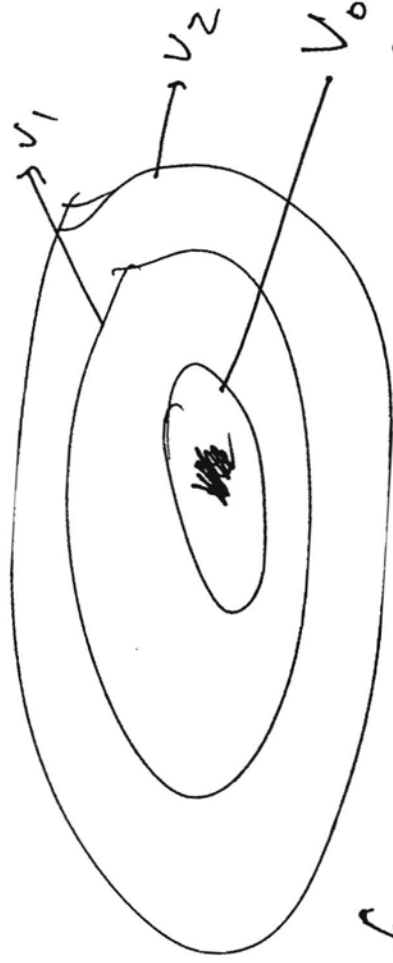
Multiresolution Analysis (Mallat)

① Scaling fn is \perp to its integer translates
(only for Haar)

② $\dots V_{-1} \subset V_0 \subset V_1 \subset V_2 \dots$

nesting of subspaces.

if $f(x) \in V_j$ then $f(2x) \in V_{j+1}$



③ Only fn common to all V_j is $f(x) \equiv 0$

$$V_{-\infty} = \{0\}$$

④ Any f_n can be represented with arbitrary precision.

$$V_{00} = \{L^2(R)\}.$$

- Can write $\Phi_{j,k}$ as linear combination of $\Phi_{j+1,k}$

$$\Phi_{j,k}(x) = \sum_n \alpha_n \Phi_{j+1,n}(x)$$

$$\Phi_{j,k}(x) = \sum_n h_{\phi}(n) \frac{j+1}{2} \phi(2x-n)$$

$$\text{Set } j=k=0 \Rightarrow \Phi_{0,0} = \phi(x)$$

$$\phi(x) = \sum_n h_{\phi}(n) \sqrt{2} \phi(2x-n)$$

— $\phi(x)$ can be built from double resolution

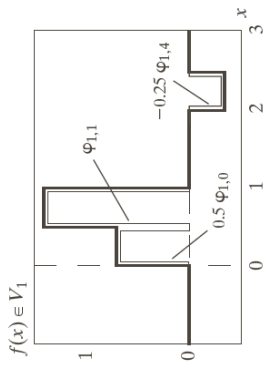
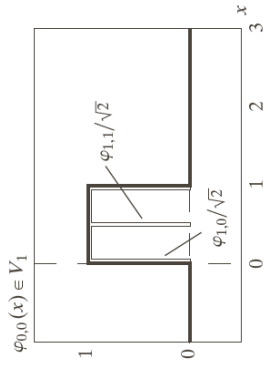
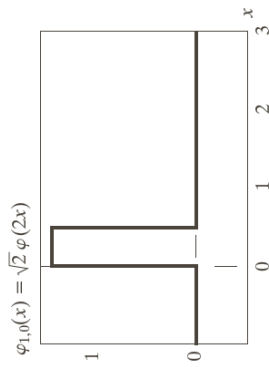
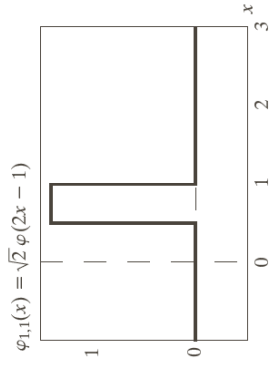
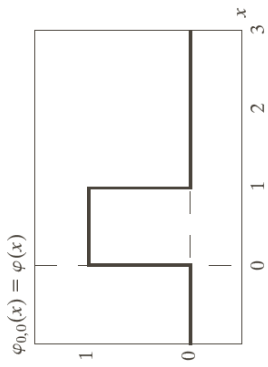
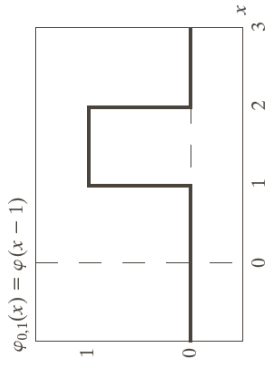
Copies of itself. i.e. from $\phi(2x)$

— Expansion f_n of V_j is linear comb of V_{j+1}

Fig 7.11f b-tw

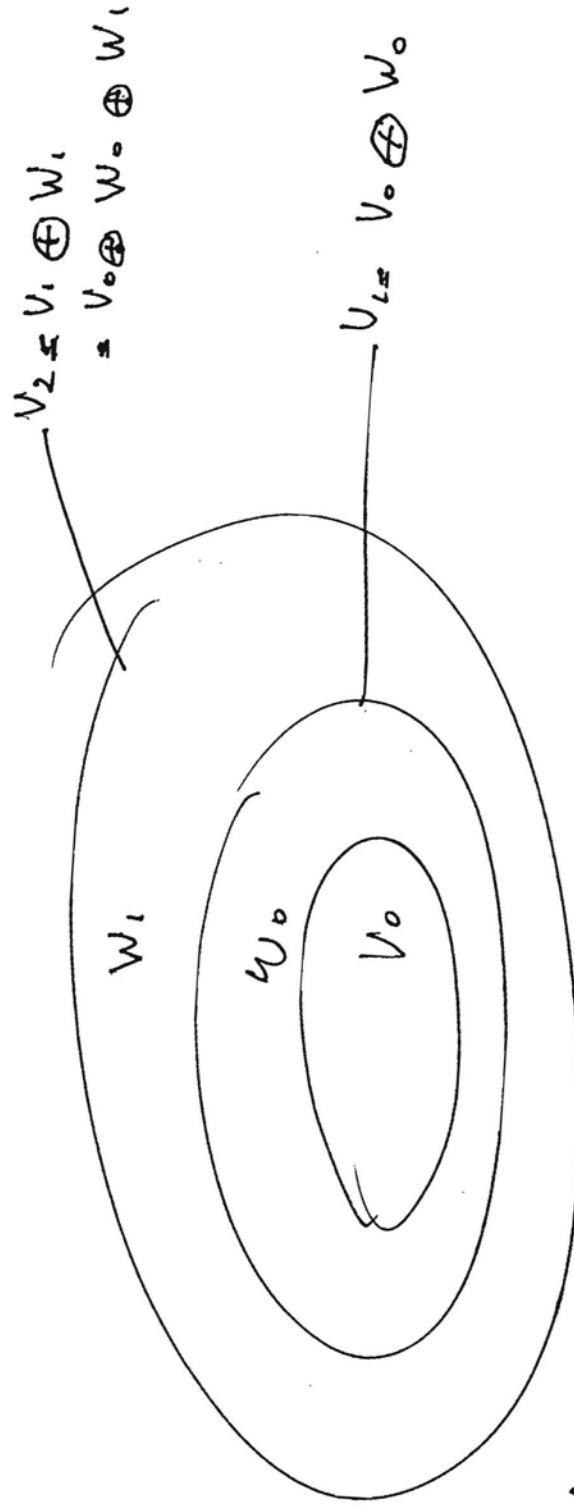
a b
c d
e f

FIGURE 7.11
Some Haar
scaling functions.



Wavelet fns

- Span The distance between 2 adjacent subspace V_j and V_{j+1}



$$\psi_{j,k}^{(x)} = 2^{j/2} \psi(2^j x - k) \quad k \in \mathbb{Z}$$

$$W_j = \text{Span}_k \{ \psi_{j,k}^{(x)} \}$$

$$V_{j+1} = V_j \oplus W_j \quad \leftarrow \text{Union of space}$$

— Orthogonal complement of V_j is V_{j+1} is W_j

$$\Rightarrow \langle \phi_{j,k}, \psi_{j,l} \rangle = 0 \quad \forall j,k,l \in \mathbb{Z}$$

$$\begin{aligned} L^2(\mathbb{R}) &= V_0 \oplus W_0 \oplus W_1 \oplus \dots \\ &= V_1 \oplus W_1 \oplus W_2 \oplus \dots \\ &= \dots \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus \\ & \quad W_1 \oplus W_2 \oplus \dots \end{aligned}$$

no need to deal with \emptyset only ϕ .



If $f \in V_0$

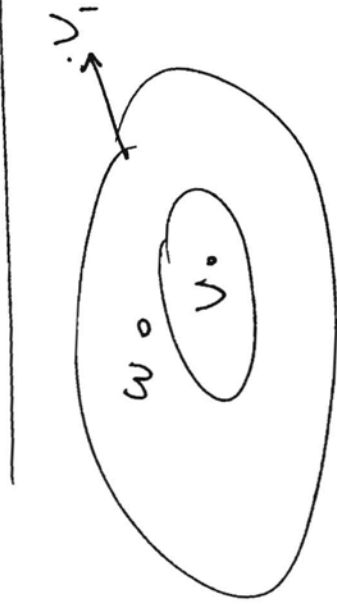
f is linear comb of scaling fn in V_0
 + linear comb of wavelet from W_0

$$L^2(\mathbb{R}) = V_j \oplus W_j \oplus W_{j+1} \oplus \dots$$

j arbitrary.

$$= V_0 \oplus W_0 \oplus W_1 \oplus W_2 + \dots$$

$$= V_5 \oplus W_6 \oplus W_7 \oplus W_8 \oplus \dots$$



Basis for $V_0 \rightarrow \phi(x)$
 $\alpha \rightarrow \phi(2x)$
 $\alpha \rightarrow \psi(x)$

$\phi(x) \subseteq$ linear comb of $\phi(2x)$

$\psi(x) \subseteq \alpha \alpha \alpha \phi(2x)$

$$\phi(x) = \sum_n h_\phi(n) \sqrt{2} \phi(2x-n)$$

Theorem by Burns $h_\psi(n) = (-1)^n h_\phi(1-n)$

\Rightarrow from $\phi(x) \rightarrow h_\phi \rightarrow h_\psi \rightarrow \psi$

⇒ For Haar

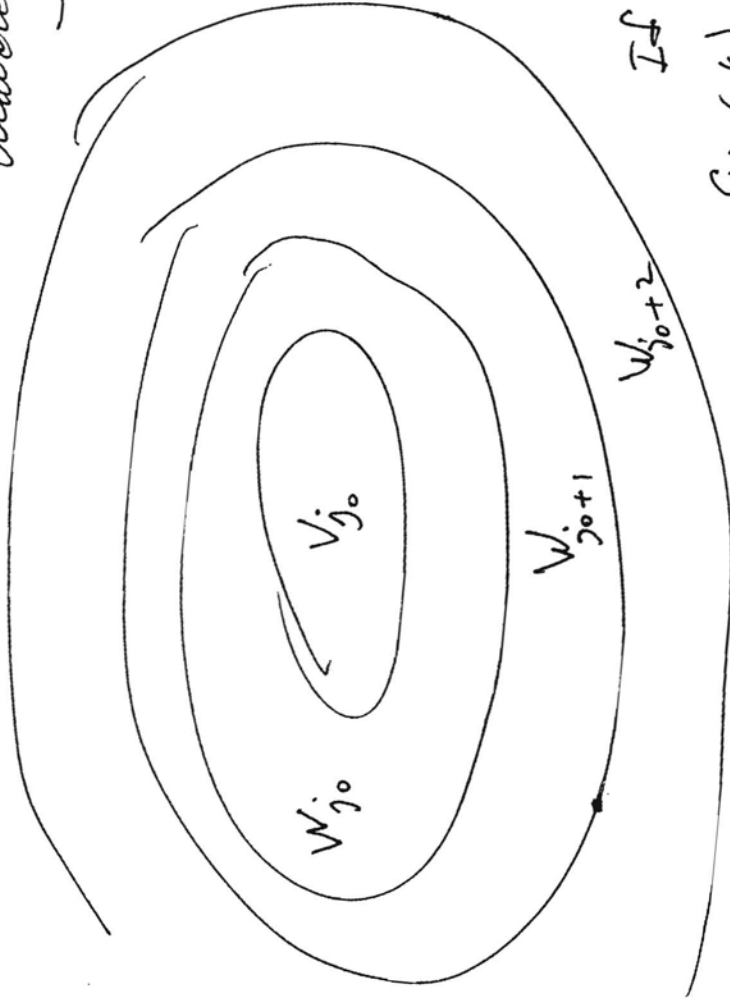
$$\psi(x) = \begin{cases} 1 & 0 \leq x < 0.5 \\ -1 & 0.5 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Show Fig 7.14 & 7.15

Wavelet Series Expansion.

Arbitrary f_0 .

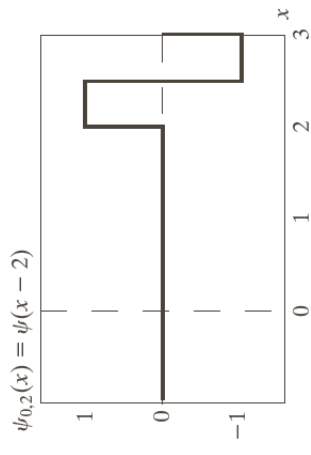
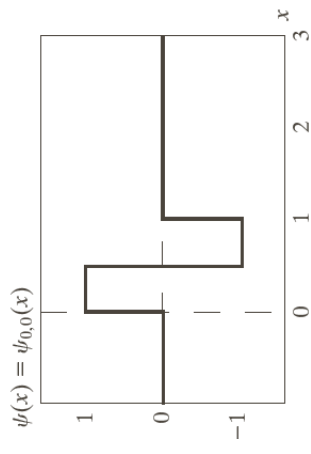
$$f(x) = \sum_k c_{j_0, k} \psi_{j_0, k}(x) + \sum_{j=j_0}^{\infty} \sum_k d_j(k) \psi_{j, k}(x)$$



If ϕ orthonormal on tight frame.

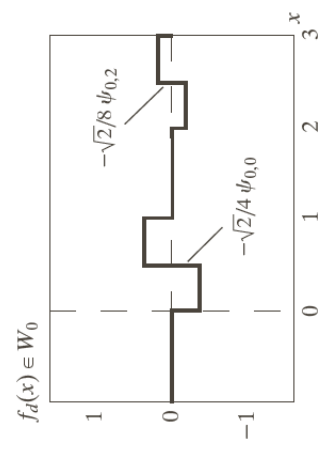
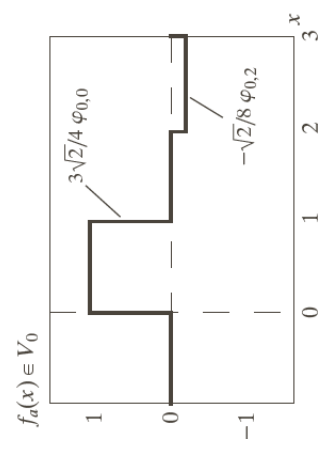
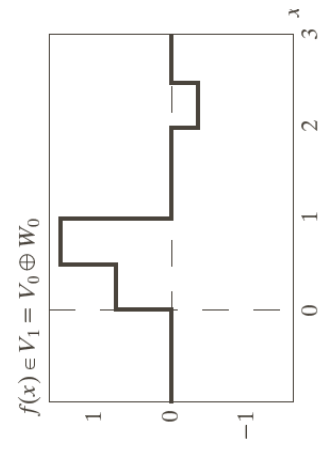
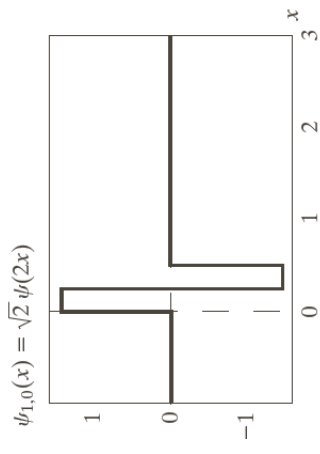
$$c_{j_0}(k) = \langle f(x), \psi_{j_0, k}(x) \rangle = \int f(x) \psi_{j_0, k}(x) dx$$

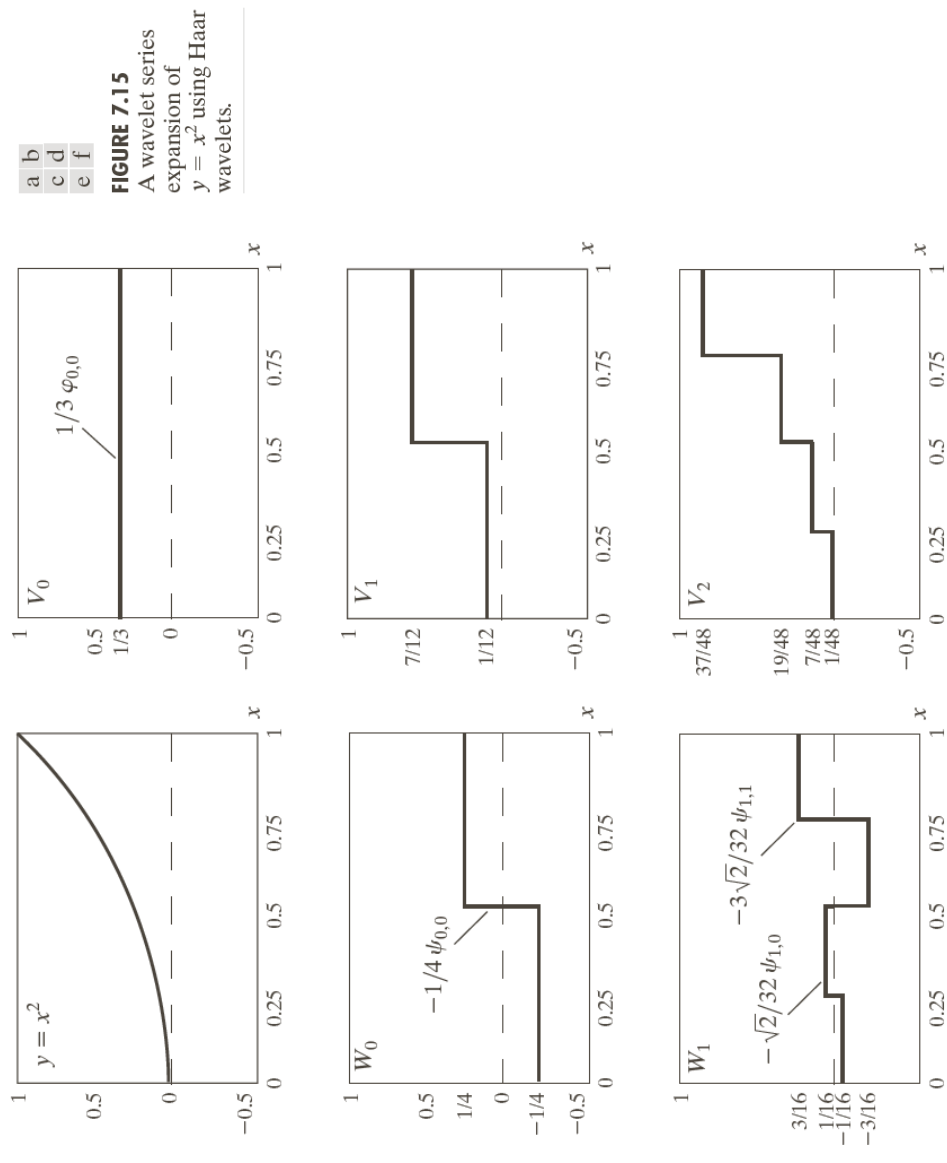
$$d_j(k) = \langle f(x), \psi_{j, k}(x) \rangle = \int f(x) \psi_{j, k}(x) dx$$



a
b
c
d
e
f

FIGURE 7.14
Haar wavelet functions in W_0 and W_1 .





a b
c d
e f

FIGURE 7.15
A wavelet series
expansion of
 $y = x^2$ using Haar
wavelets.

Show Fig 7.15

Discrete Wavelet Transform

So far ~~deal~~ dealt with $f(x)$ x Real.

now deal with $f(n)$ n integers \Rightarrow square root f_n .

Forward DWT coefficients for $f(n)$, (assuming Tight frame or orthogonal) $\phi_{j_0, k}(n)$ sampled version of $\phi_{j_0, k}(x)$

$$W_\phi(j_0, k) = \frac{1}{\sqrt{M}} \sum_n f(n) \phi_{j_0, k}(n) \quad j \geq j_0$$

$$W_\phi(j, k) = \frac{1}{\sqrt{M}} \sum_n f(n) \phi_{j, k}(n)$$

Then $f(n) = \frac{1}{\sqrt{M}} \sum_k W_\phi(j_0, k) \phi_{j_0, k}(n) +$

$$\frac{1}{\sqrt{M}} \sum_{j=j_0}^j \sum_k W_\phi(j, k) \phi_{j, k}(n)$$

Continuous Wavelet Transform

- Already discussed last time

$$W_{\psi}(s, \tau) = \int_{-\infty}^{+\infty} f(x) \psi_{s, \tau}(x) dx$$

$$\psi_{s, \tau}(x) = \frac{1}{\sqrt{s}} \psi\left(\frac{x - \tau}{s}\right)$$

$s = \text{scale}$ $\tau = \text{translation}$

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} |W_{\psi}(s, \tau)|^2 \frac{d\tau ds}{s^2} = \int_{-\infty}^{+\infty} |f(x)|^2 dx$$

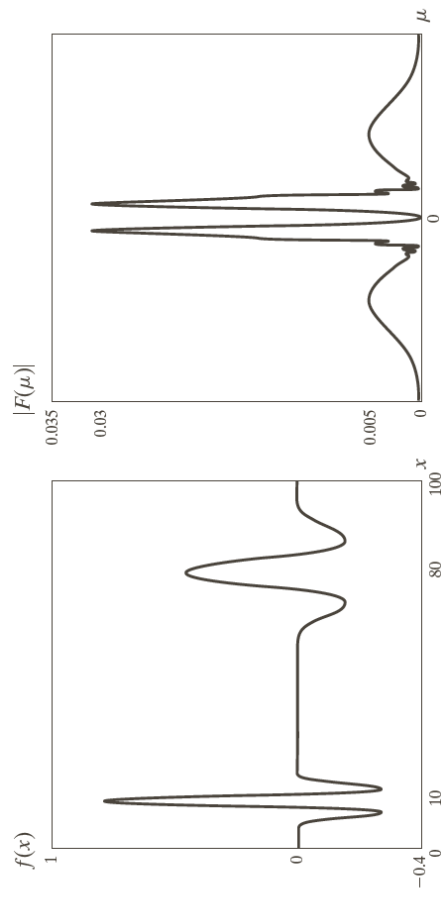
Inverse Wavelet $f(x) = \frac{1}{C_{\psi}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\mu)|^2}{|\mu|} d\mu \leftarrow \text{admissibility criterion.}$

where $C_{\psi} = \int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\mu)|^2}{|\mu|} d\mu \leftarrow \text{admissibility criterion.}$

$$s \propto \frac{1}{2^j}$$

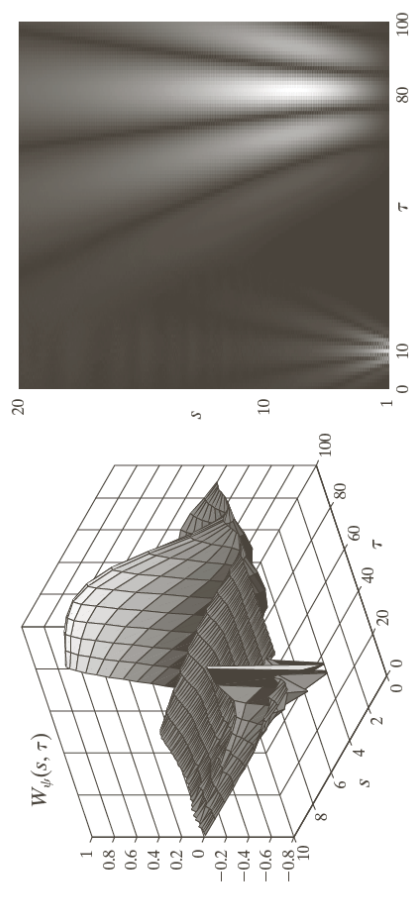
- compression - dilation

- Show Fig 7.16 6+10



a	b
c	d

FIGURE 7.16
 The continuous wavelet transform (c and d) and Fourier spectrum (b) of a continuous 1-D function (a).



Fast Wavelet Transform

$$\phi(x) = \sum_n h_\phi(n) \sqrt{2} \phi(2x-n)$$

$$x \leftarrow 2^j$$

$$\begin{aligned} \phi(2^j x - k) &= \sum_n h_\phi(n) \sqrt{2} \phi(2^j x - k - n) \\ &= \sum_m h_\phi(m - 2k) \sqrt{2} \phi(2^{j+1} x - m) \end{aligned}$$

Similarly

$$\psi(2^j x - k) = \sum_m h_\psi(m - 2k) \sqrt{2} \psi(2^{j+1} x - m) \quad \Rightarrow$$

Recall $d_j(k) = \int f(x) 2^{j/2} \psi(2^j x - k) dx$

$$d_j(k) = \sum_m h_\psi(m - 2k) c_{j+1}(m)$$

Similarly

$$c_j(k) = \sum_m h_\phi(m - 2k) c_{j+1}(m)$$

$$\begin{aligned}
 c_j(k) &\longrightarrow W_\psi(j,k) \\
 d_j(k) &\longrightarrow W_\phi(j,k)
 \end{aligned}
 \left. \vphantom{\begin{aligned} c_j(k) \\ d_j(k) \end{aligned}} \right\} \text{DWT}$$

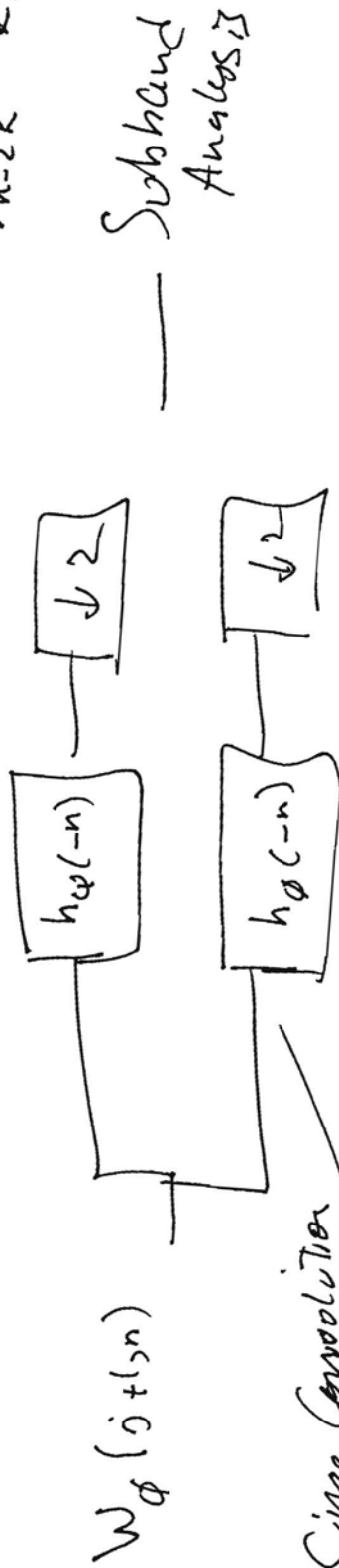
as $f(x) \longrightarrow f(n)$

Then

$$W_\psi(j,k) = \sum_m h_\psi(m-2k) W_\phi(j+1,m)$$

$$W_\phi(j,k) = \sum_m h_\phi(m-2k) W_\phi(j+1,m)$$

$$\Rightarrow \begin{cases} W_\psi(j,k) = h_\psi(-n) * W_\phi(j+1,n) \\ W_\phi(j,k) = h_\phi(-n) * W_\phi(j+1,n) \end{cases} \left. \vphantom{\begin{cases} W_\psi(j,k) \\ W_\phi(j,k) \end{cases}} \right\} \begin{array}{l} n=2k \quad k \geq 0 \\ n=2k \quad k > 0 \end{array}$$



Show Fig 7.18

a
b

FIGURE 7.18

(a) A two-stage or two-scale FWT analysis bank and (b) its frequency splitting characteristics.

