Systems

- system Transform input into output

$$T\left[\underbrace{x(n_1, n_2)}_{\text{input}}\right] = \underbrace{y(n_1, n_2)}_{\text{output}}$$

- Linear: $T\{a\, x_1(n_1, n_2) + b\, x_2(n_1, n_2)\}$

$$= a\, T\{x_1(n_1, n_2)\} + b\, T\{x_2(n_1, n_2)\}$$

- Shift Invariance:

If $T\{x(n_1, n_2)\} = y(n_1, n_2)$

then $T[x(n_1 - k_1, n_2 - k_2)] = y(n_1 - k_1, n_2 - k_2)$

2D- LSI system.

LSI $\longrightarrow$ Impulse response.

$h(n_1, n_2)$

- enables to entirely characterize an LSI system.
- tells you output given input.

$$Y(n_1, n_2) = X * h$$

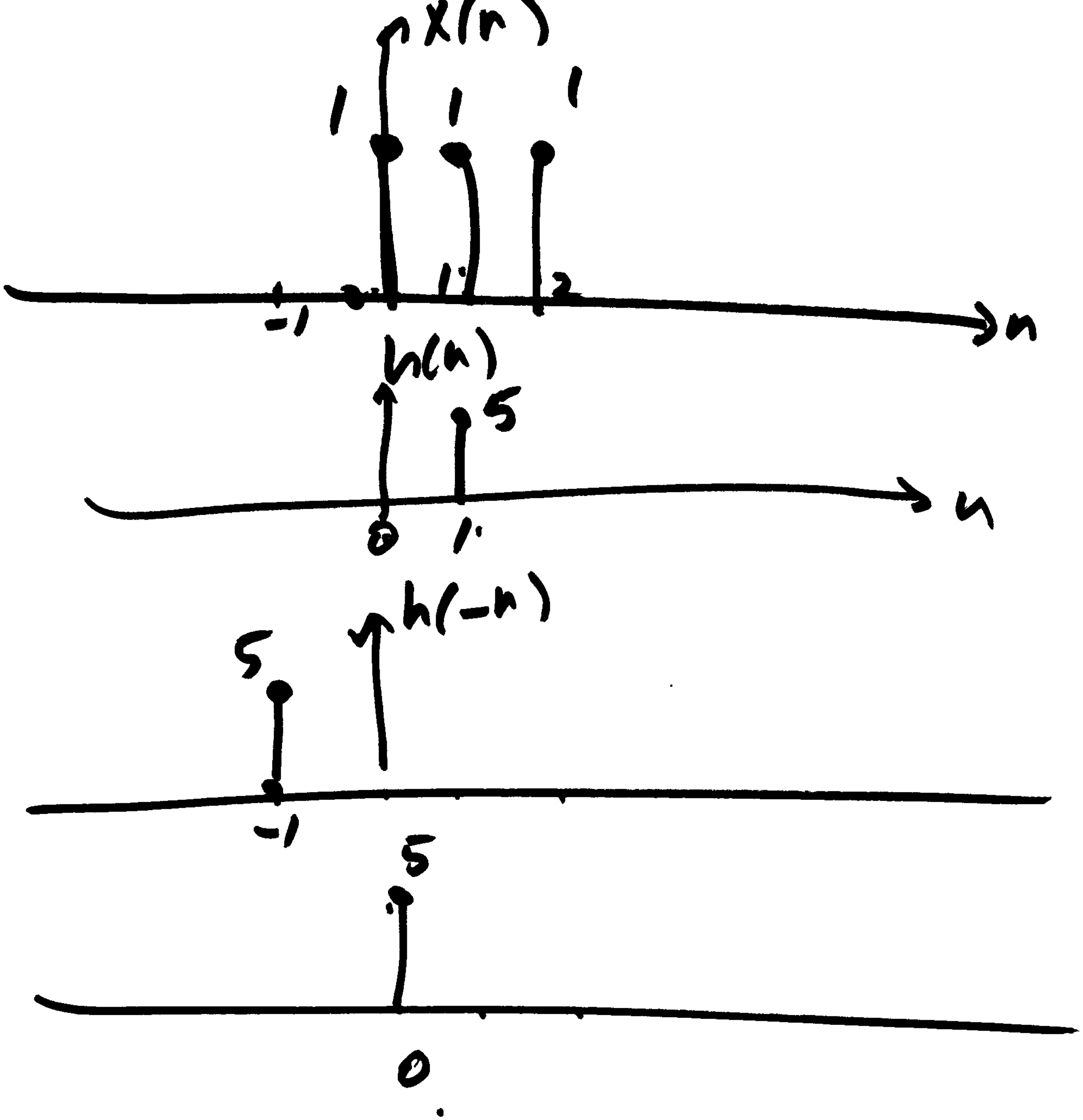
$*$ $\longrightarrow$ Convolution

$$y(n_1, n_2) = \sum_{k_1} \sum_{k_2} X(k_1, k_2)\, h(n_1 - k_1, n_2 - k_2)$$

2D

$$Y(n) = \sum_{k} X(k)\, h(n - k)$$

1D



$Y(0) = 0$

$Y(1) = 5$

Separable LSI system

$$h(n_1, n_2) = h_1(n_1)\, h_2(n_2)$$

$x(n_1, n_2)$ $N \times N$ $\quad x \rightarrow \boxed{h} \rightarrow y.$

h : $M \times M.$

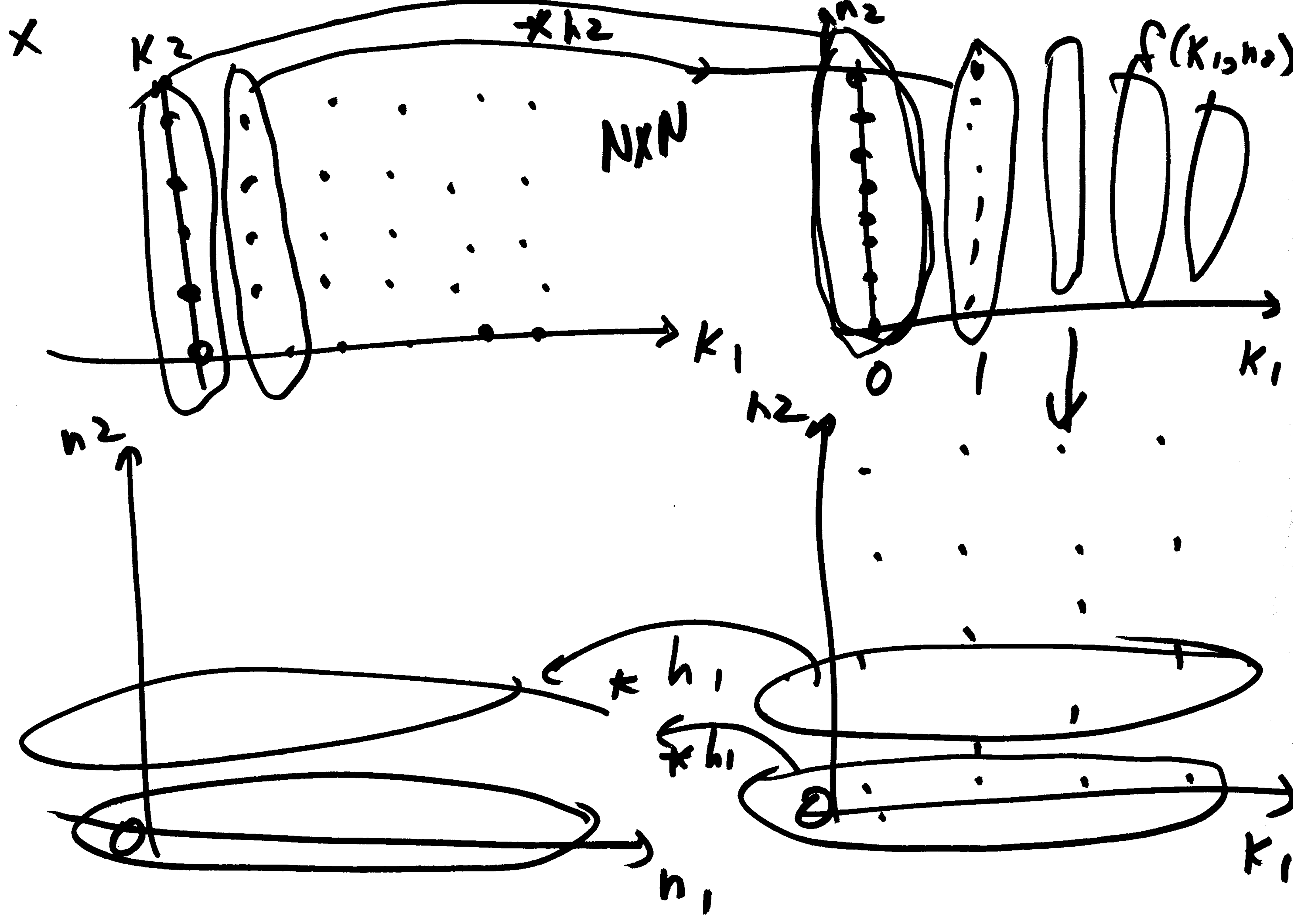
general:

$$y(n_1, n_2) = \sum_{k_1} \sum_{k_2} x(k_1, k_2)\, h(n_1 - k_1, n_2 - k_2)$$

sep. sys.

$$= \sum_{k_1} \sum_{k_2} x(k_1, k_2)\, h_1(n_1 - k_1)\, h_2(n_2 - k_2)$$

$$= \sum_{k_1} h_1(n_1 - k_1) \underbrace{\sum_{k_2} x(k_1, k_2)\, h_2(n_2 - k_2)}_{f(k_1, n_2) = x(k_1) * h_2}$$


X
k2
*h2
NXN
k1
n2
f(k1,n2)
0
1
k1
n2
n2
*h1
*h1
n1
k1

Summary

1. Series of 1D convolution To get $f(k_1, n_2)$:

N columns $\longrightarrow$ N convolution. NM oper / convolut

$N^2 M$ op To build f

(2) Series of 1-D conv To get $y(n_1, n_2)$

N rows $\longrightarrow$ N convolution. NM op / conval

$N^2 M$

Total $2 N^2 M$.

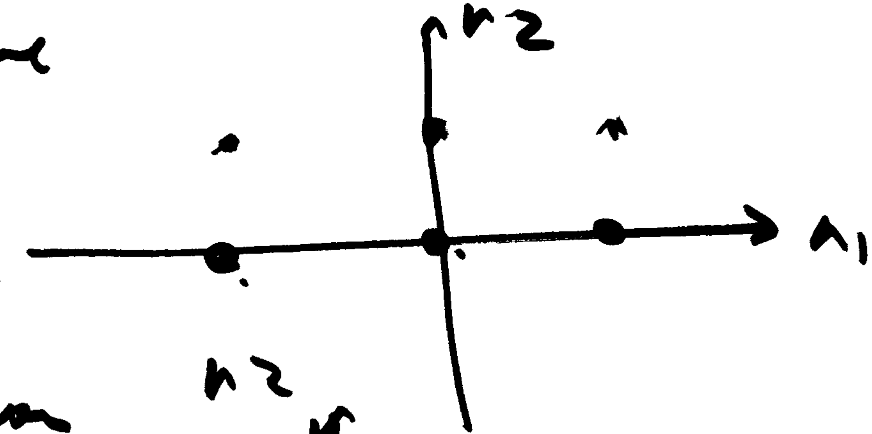
LSI sytem

* impulse response. $\rightarrow h(n_1, n_2)$
* stability of LSI system.
* BIBO stability $\rightarrow$

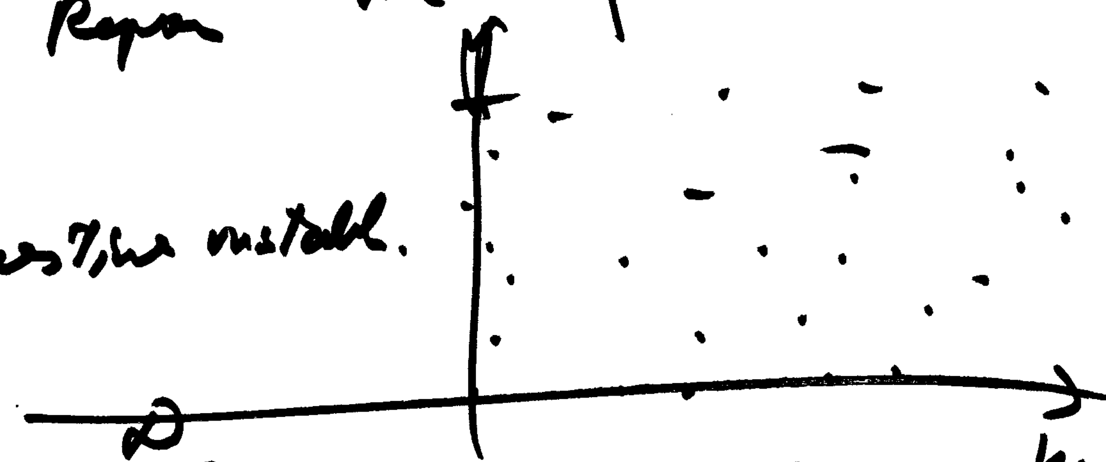
  bounded input, bounded output.
* Can show, necess + suff. cond. $h(n_1, n_2)$ to be BIBO stable.

$$\sum_{n_1 = -\infty}^{+\infty} \sum_{n_2} |h(n_1, n_2)| < \infty$$

Finite impulse response
(FIR).
$\Longrightarrow$ Always stable.



Infinite impulse Respon
(IIR)
some stable, sometimes unstable.

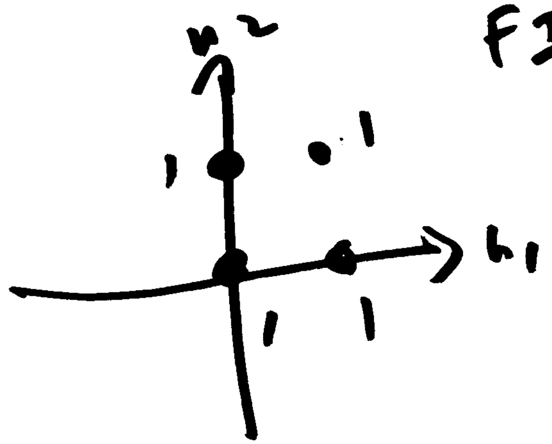


$\sum_{n=1}^{\infty} \frac{1}{n}$ blows up $\sum_{n=1}^{\infty} \frac{1}{n^2}$ does not blow up.

$h(n_1, n_2) \xrightarrow{Z.T.} \sum_{n_1} \sum_{n_2} h(n_1, n_2) z_1^{-n_1} z_2^{-n_2}$

FIR

$= H(z_1, z_2) =$ 2D polynomial in z_1 & z_2



FIR

$$H(z_1, z_2) = 1 + z_1^{-1} + z_2^{-1} + z_1^{-1} z_2^{-1}$$

2D polynomial in z_1^{-1} and in z_2^{-1}

2.D.F.T.

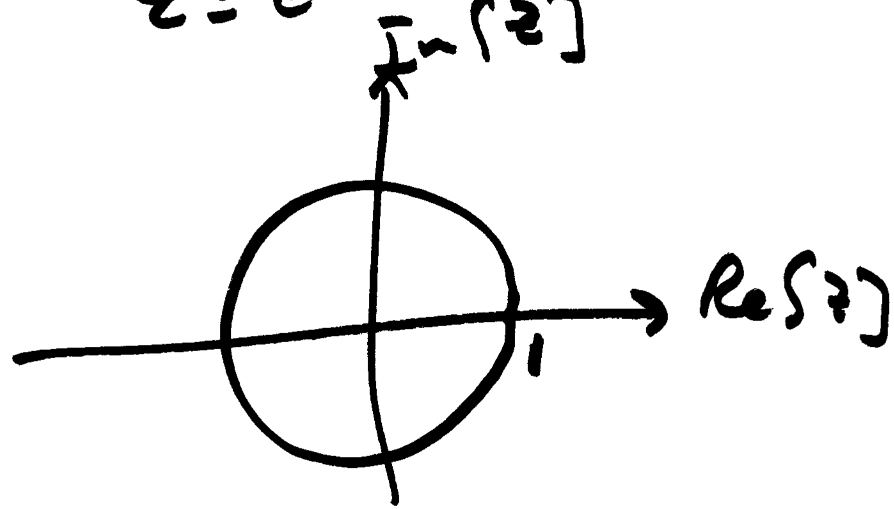
$$H(\omega_1, \omega_2) = \sum_{n_1} \sum_{n_2} h(n_1, n_2)\, e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

$$e^{j\omega_0 n} \longrightarrow \boxed{\begin{matrix} \text{LSI} \\ h(n) \end{matrix}} \longrightarrow \underbrace{H(\omega_0)}_{\text{not a fn of } n.} e^{j\omega_0 n}$$

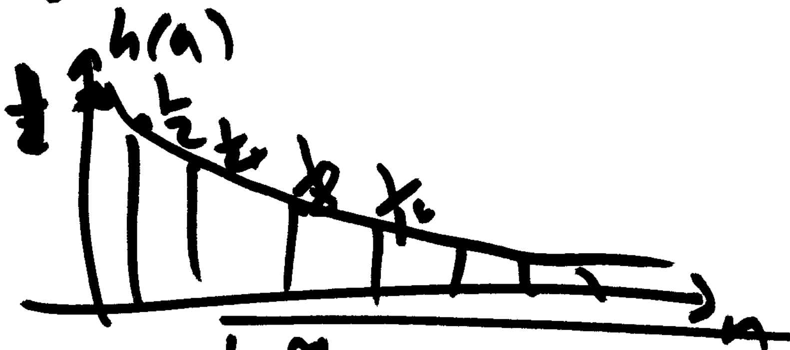
$$H(\omega_0) = \left[H(\omega)\right]_{\omega=\omega_0} = \left[\sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n}\right]_{\omega=\omega_0}$$

$$\left.\begin{aligned} H(\omega) &= \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} \\ H(z) &= \sum_{n=-\infty}^{+\infty} h(n)\, z^{-n} \end{aligned}\right\} \Longrightarrow$$

$$H(\omega) = \Big[H(z) \Big]_{z = e^{j\omega}}$$



$$H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} \longrightarrow h(n) = \cdot \; - \; - \cdots$$



ROC . $\}$ outside circl

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

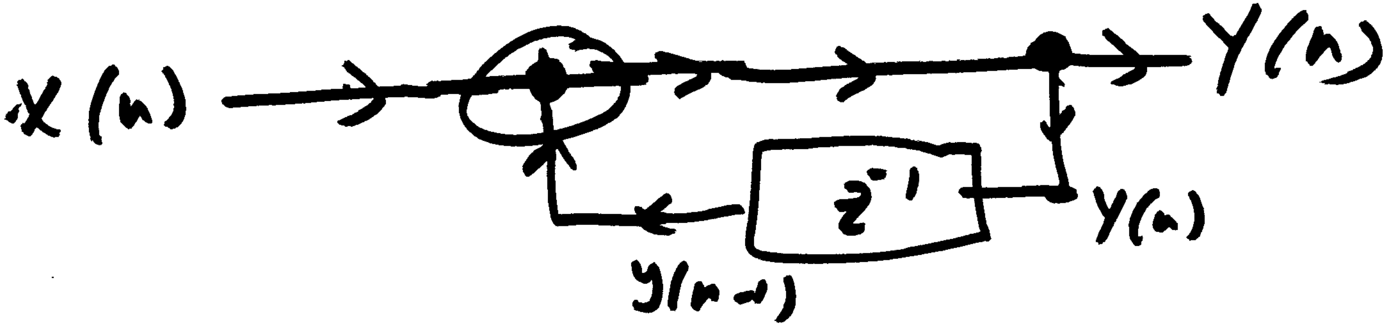
$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

BIBO

$$Y(z)\left[1 - \frac{1}{2} z^{-1}\right] = X(z).$$

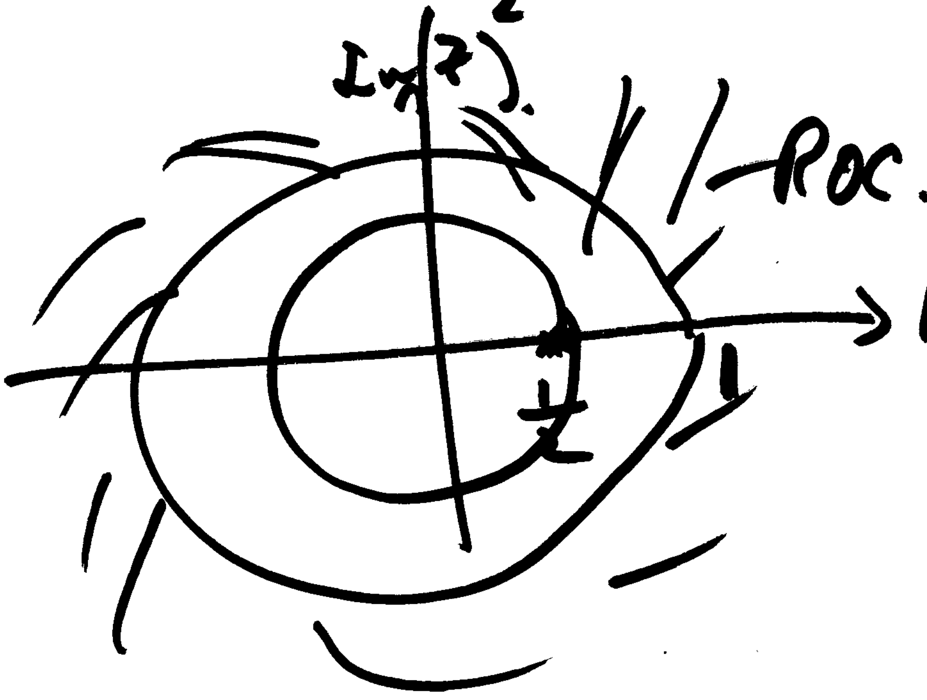
$$y(n) - \frac{1}{2} y(n-1) = x(n)$$

$$y(n) = x(n) + \frac{1}{2} y(n-1)$$



pole $1-\frac{1}{2}z^{-1}=0 \Rightarrow 1=\frac{1}{2}z^{-1}$

$2=z^{-1} \Rightarrow z=\frac{1}{2}$

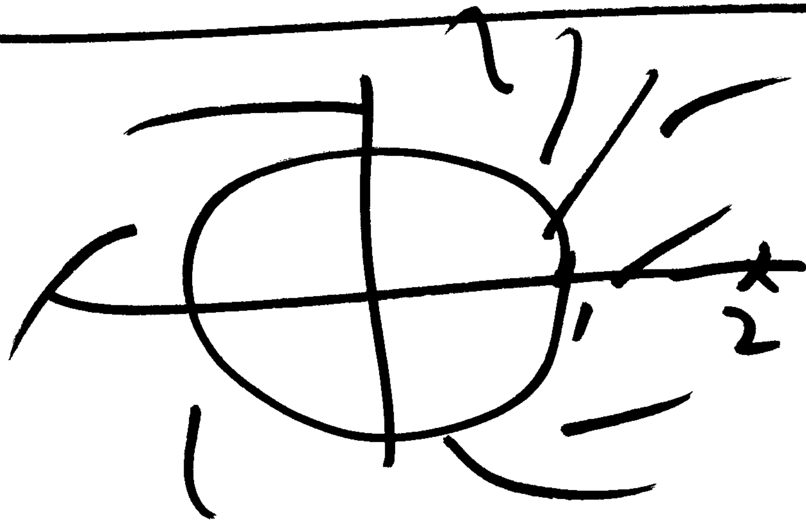


ROC.

$|z|>\frac{1}{2}$

$h(n)=\frac{1}{2}^{n}u(n)$

Unstable



$h(n)=2^{n}u(n)$

$=1+2+4+8+$

$H(z)=\frac{1}{1-2z^{-1}}$

$$H(z) = \frac{1}{1+5z^{-1}+3z^{-2}+9z^{-3}+12z^{-4}+18z^{-5}}$$

2D polynomial can always **Be** factored.

→ Fundamental Thm of Algebra.

Any 1-D polynomial ~~[illegible]~~ of deg. n can be factored into n polynomials of deg. 1.

$$= \frac{1}{\prod_{i=1}^{5} (1-\alpha_i z^{-1})}$$

Story in 2D

IIR: $$H(z_1, z_2) = \frac{1}{P(z_1^{-1}, z_2^{-1})}$$

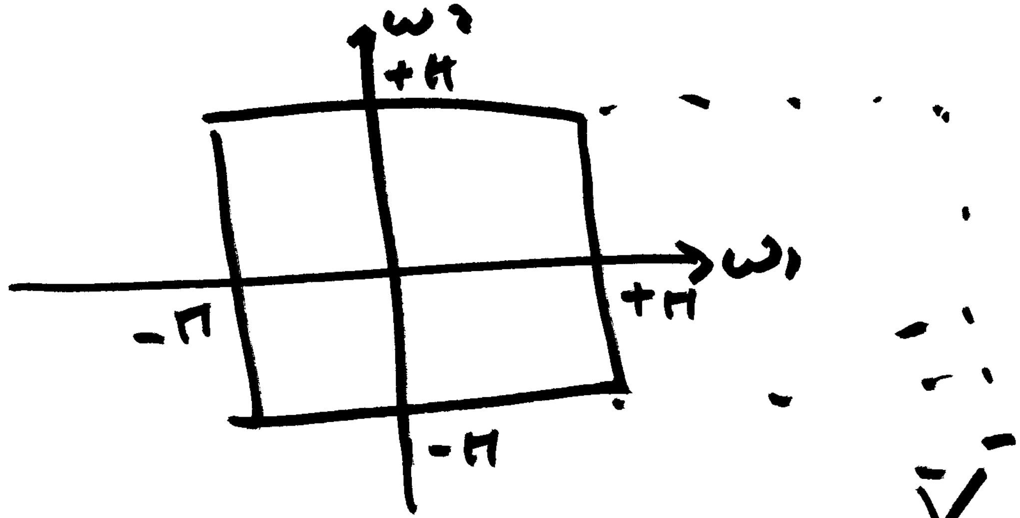
$$P(z_1^{-1}, z_2^{-1}) = (1 - z_1^{-1})(1 - z_2^{-1})$$

Thm: Haya 1980s: Set of reducible 2D polynomials is of measure 0 in the set of all 2D polynomials.

2D → unit bisurface.

Do The pde surface cross The unit Bi-surface?. $\Longrightarrow$ some Thms covered later..

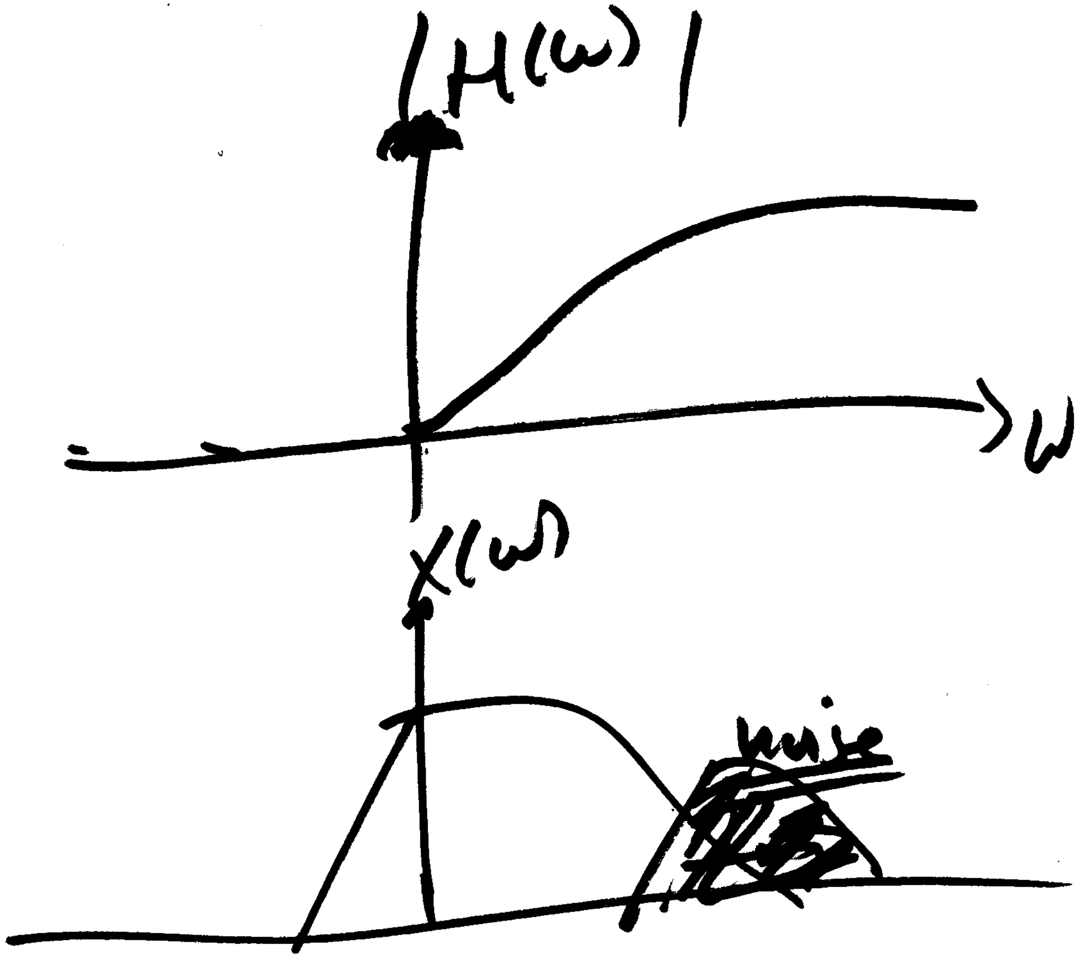
Proving stability in 2D is hard



$x(n_1, n_2) \longrightarrow \bar{X}(\omega_1, \omega_2)$

discrete integer

Real variable.



$|H(\omega)|$
ω
$X(\omega)$
noise

Signal.