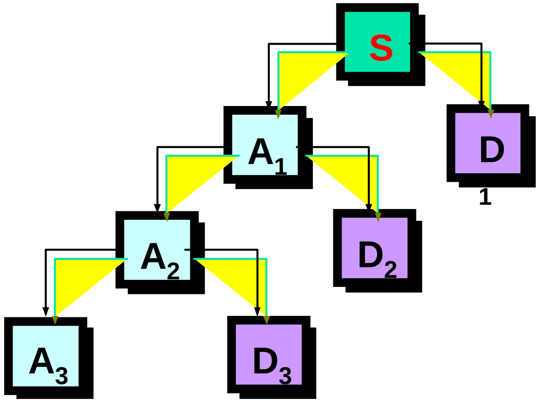
Introduction to Wavelet



Bhushan D Patil
PhD Research Scholar
Department of Electrical Engineering
Indian Institute of Technology, Bombay
Powai, Mumbai. 400076

Outline of Talk

- Overview
- Historical Development
- Time vs Frequency Domain Analysis
- Fourier Analysis
- Fourier vs Wavelet Transforms
- Wavelet Analysis
- Typical Applications
- References

OVERVIEW

- Wavelet
  - A small wave
- Wavelet Transforms
  - Convert a signal into a series of wavelets
  - Provide a way for analyzing waveforms, bounded in both frequency and duration
  - Allow signals to be stored more efficiently than by Fourier transform
  - Be able to better approximate real-world signals
  - Well-suited for approximating data with sharp discontinuities
- “The Forest & the Trees”
  - Notice gross features with a large "window“
  - Notice small features with a small

Historical Development

- Pre-1930
  - Joseph Fourier (1807) with his theories of frequency analysis
- The 1930^s
  - Using scale-varying basis functions; computing the energy of a function
- 1960-1980
  - Guido Weiss and Ronald R. Coifman; Grossman and Morlet
- Post-1980
  - Stephane Mallat; Y. Meyer; Ingrid Daubechies; wavelet applications today

Mathematical Transformation

- Why
  - To obtain a further information from the signal that is not readily available in the raw signal.
- Raw Signal
  - Normally the time-domain signal
- Processed Signal
  - A signal that has been "transformed" by any of the available mathematical transformations
- Fourier Transformation
  - The most popular transformation

FREQUENCY ANALYSIS

- Frequency Spectrum
  - Be basically the frequency components (spectral components) of that signal
  - Show what frequencies exists in the signal
- Fourier Transform (FT)
  - One way to find the frequency content
  - Tells how much of each frequency exists in a signal

$$X(k+1) = \sum_{n=0}^{N-1} x(n+1) \cdot W_N^{kn}$$

$$x(n+1) = \frac{1}{N} \sum_{k=0}^{N-1} X(k+1) \cdot W_N^{-kn}$$

$$w_N = e^{-j\left(\frac{2\pi}{N}\right)}$$

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-2j\pi ft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{2j\pi ft} df$$

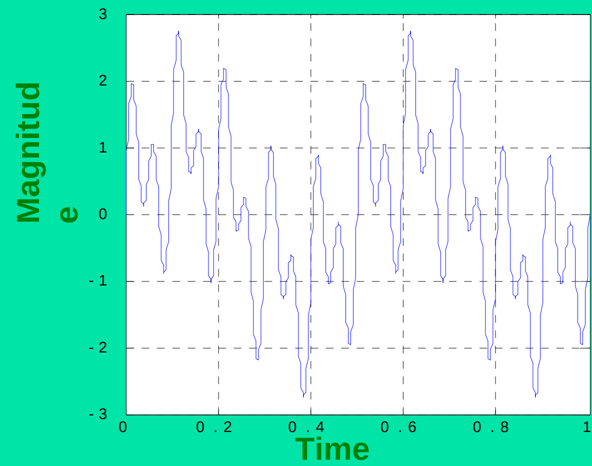
STATIONARITY OF SIGNAL

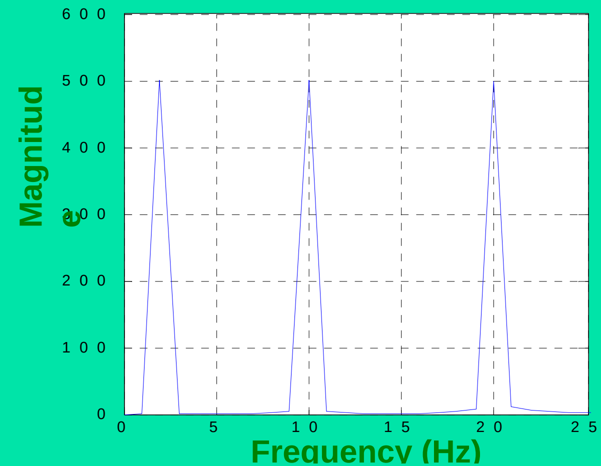
- Stationary Signal
  - Signals with frequency content unchanged in time
  - All frequency components exist at all times
- Non-stationary Signal
  - Frequency changes in time
  - One example: the "Chirp Signal"

STATIONARITY OF SIGNAL

2 Hz + 10 Hz + 20Hz

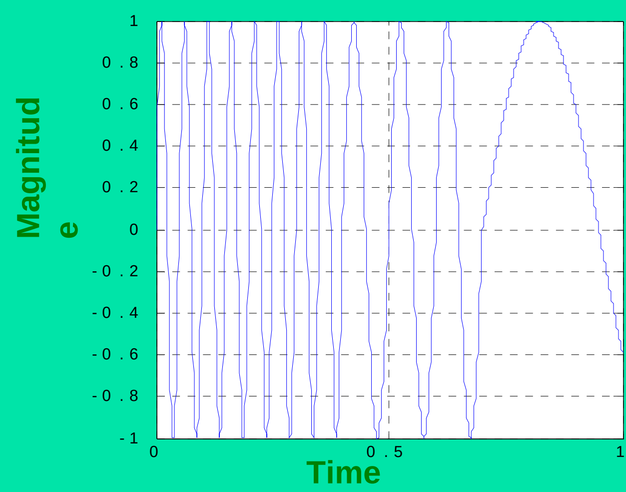
Stationary

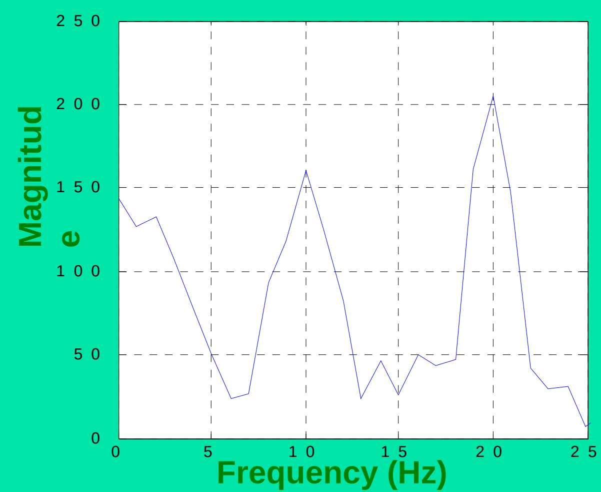




0.0-0.4: 2 Hz +
0.4-0.7: 10 Hz +
0.7-1.0: 20Hz

Non-Stationary



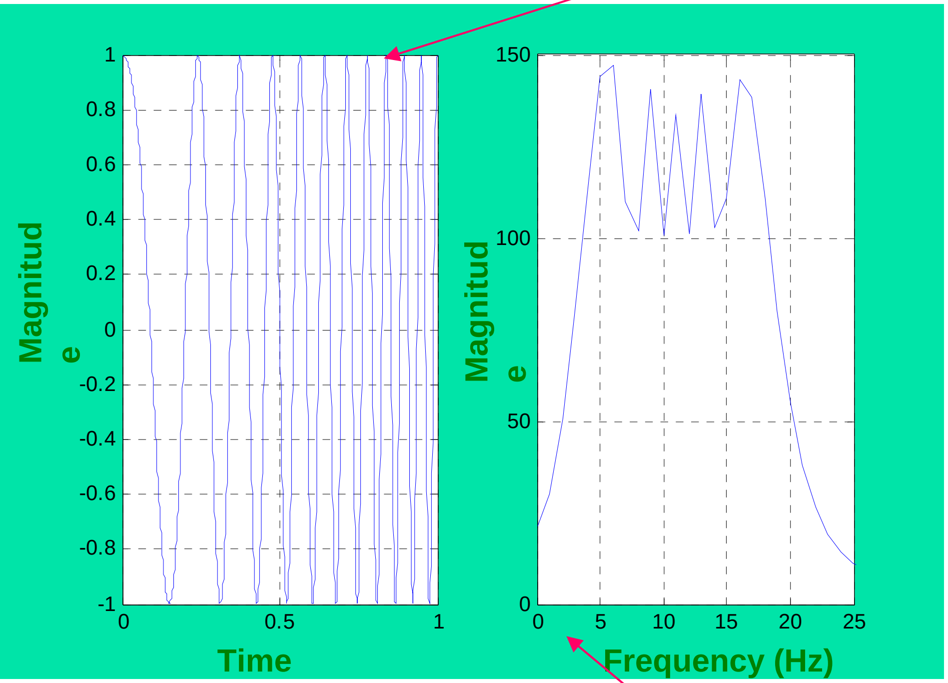


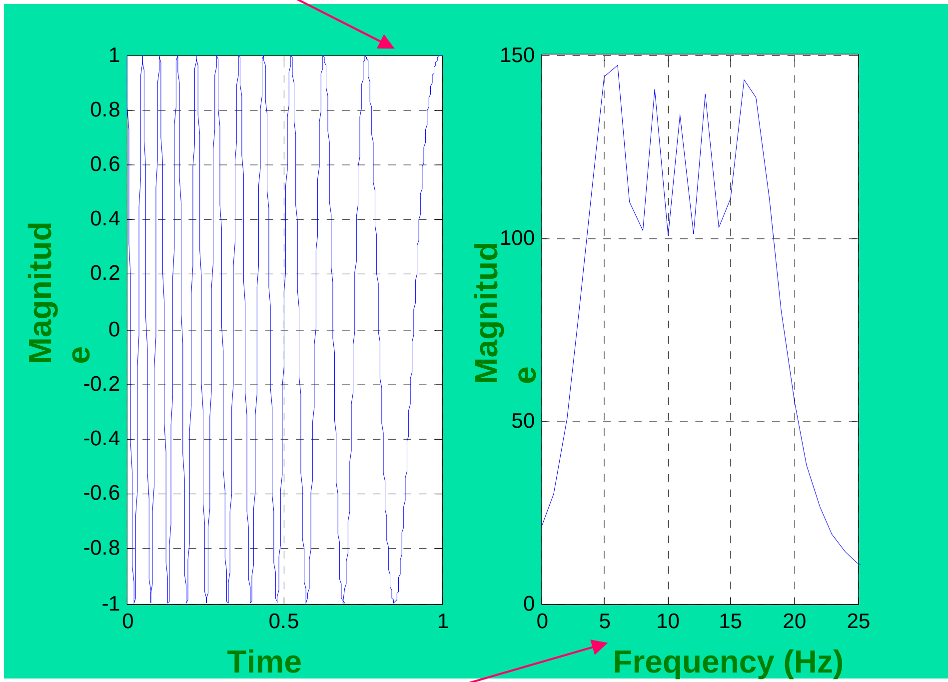
CHIRP SIGNALS

Frequency: 2 Hz to 20 Hz

Different in Time Domain

Frequency: 20 Hz to 2 Hz





Same in Frequency Domain

At what time the frequency components occur? FT can not tell!

NOTHING MORE, NOTHING LESS

- FT Only Gives what Frequency Components Exist in the Signal
- The Time and Frequency Information can not be Seen at the Same Time
- Time-frequency Representation of the Signal is Needed

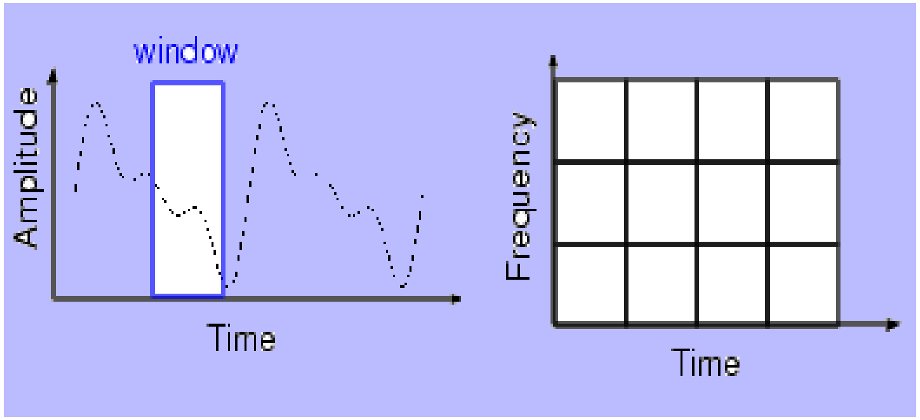
Most of Transportation Signals are Non-stationary.

(We need to know whether and also when an incident was happened.)

ONE EARLIER SOLUTION: SHORT-TIME FOURIER TRANSFORM (STFT)

SFORT TIME FOURIER TRANSFORM (STFT)

- Dennis Gabor (1946) Used STFT
  - To analyze only a small section of the signal at a time -- a technique called *Windowing the Signal*.
- The Segment of Signal is Assumed *Stationary*
- A 3D transform



$$\mathrm{STFT}_X^{(\omega)}(t', f) = \int_t \left[x(t) \bullet \omega^*(t - t')\right] \bullet e^{-j2\pi ft}\,dt$$

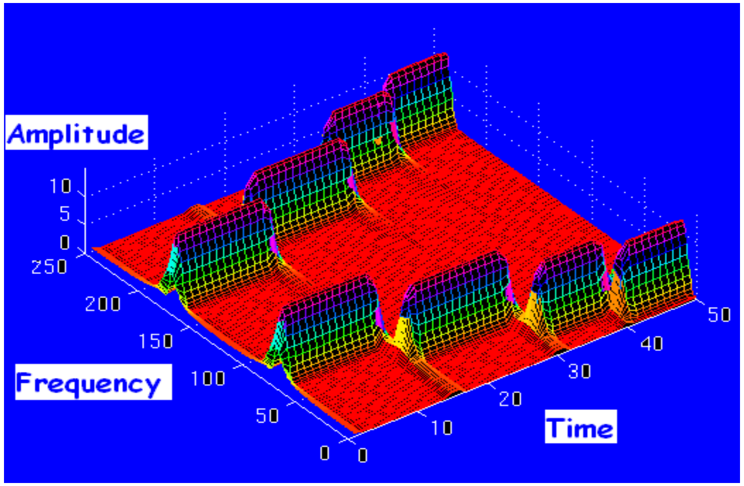
$\omega(t)$: the window function

A function of time and frequency

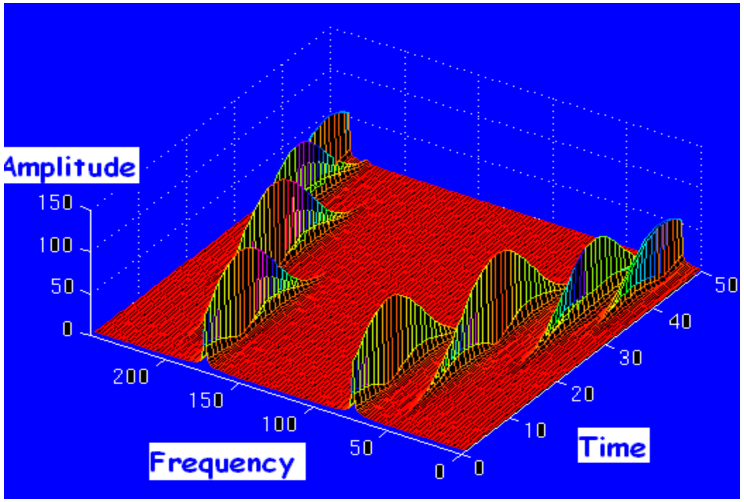
DRAWBACKS OF STFT

- Unchanged Window
- Dilemma of Resolution
  - Narrow window -> poor frequency resolution
  - Wide window -> poor time resolution
- Heisenberg Uncertainty Principle
  - Cannot know what frequency exists at what time intervals

Via Narrow Window



Via Wide Window



MULTIRESOLUTION ANALYSIS (MRA)

- Wavelet Transform
  - An alternative approach to the short time Fourier transform to overcome the resolution problem
  - Similar to STFT: signal is multiplied with a function
- Multiresolution Analysis
  - Analyze the signal at different frequencies with different resolutions
  - Good time resolution and poor frequency resolution at high frequencies
  - Good frequency resolution and poor time resolution at low frequencies
  - More suitable for short duration of higher frequency; and longer duration of lower frequency components

PRINCIPLES OF WAELET TRANSFORM

- Split Up the Signal into a Bunch of Signals
- Representing the Same Signal, but all Corresponding to Different Frequency Bands
- Only Providing What Frequency Bands Exists at What Time Intervals

DEFINITION OF CONTINUOUS WAVELET TRANSFORM

$$CWT_x^{\psi}(\tau, s) = \Psi_x^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t) \bullet \psi^*\left(\frac{t-\tau}{s}\right) dt$$

Translation

(The location of the window)

Scale

Mother Wavelet

- Wavelet
  - Small wave
  - Means the window function is of finite length
- Mother Wavelet
  - A prototype for generating the other window functions
  - All the used windows are its dilated or compressed and shifted versions

SCALE

- Scale
  - S>1: dilate the signal
  - S<1: compress the signal
- Low Frequency -> High Scale -> Non-detailed Global View of Signal -> Span Entire Signal
- High Frequency -> Low Scale -> Detailed View Last in Short Time
- Only Limited Interval of Scales is Necessary

COMPUTATION OF CWT

$$\mathrm{CWT}_x^{\psi}(\tau, s) = \Psi_x^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t) \bullet \psi^{*}\left(\frac{t-\tau}{s}\right) dt$$

Step 1: The wavelet is placed at the beginning of the signal, and set s=1 (the most compressed wavelet);

Step 2: The wavelet function at scale "1" is multiplied by the signal, and integrated over all times; then multiplied by ;

Step 3: Shift the wavelet to *t*= , and get the transform value at *t*= and *s*=1;

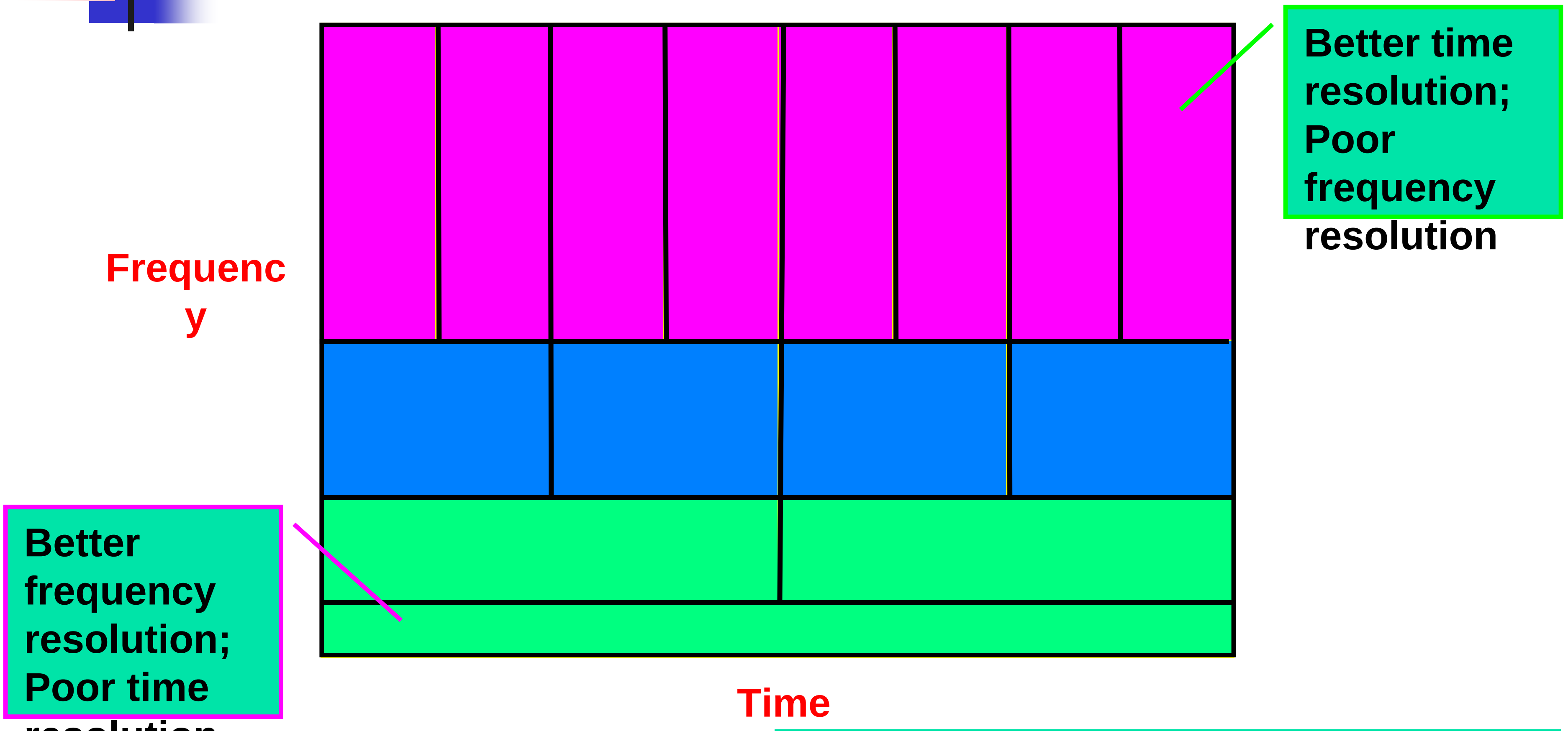
Step 4: Repeat the procedure until the wavelet reaches the end of the signal;

Step 5: Scale s is increased by a sufficiently small value, the above procedure is repeated for all *s;*

Step 6: Each computation for a given *s* fills the single row of the time-scale plane;

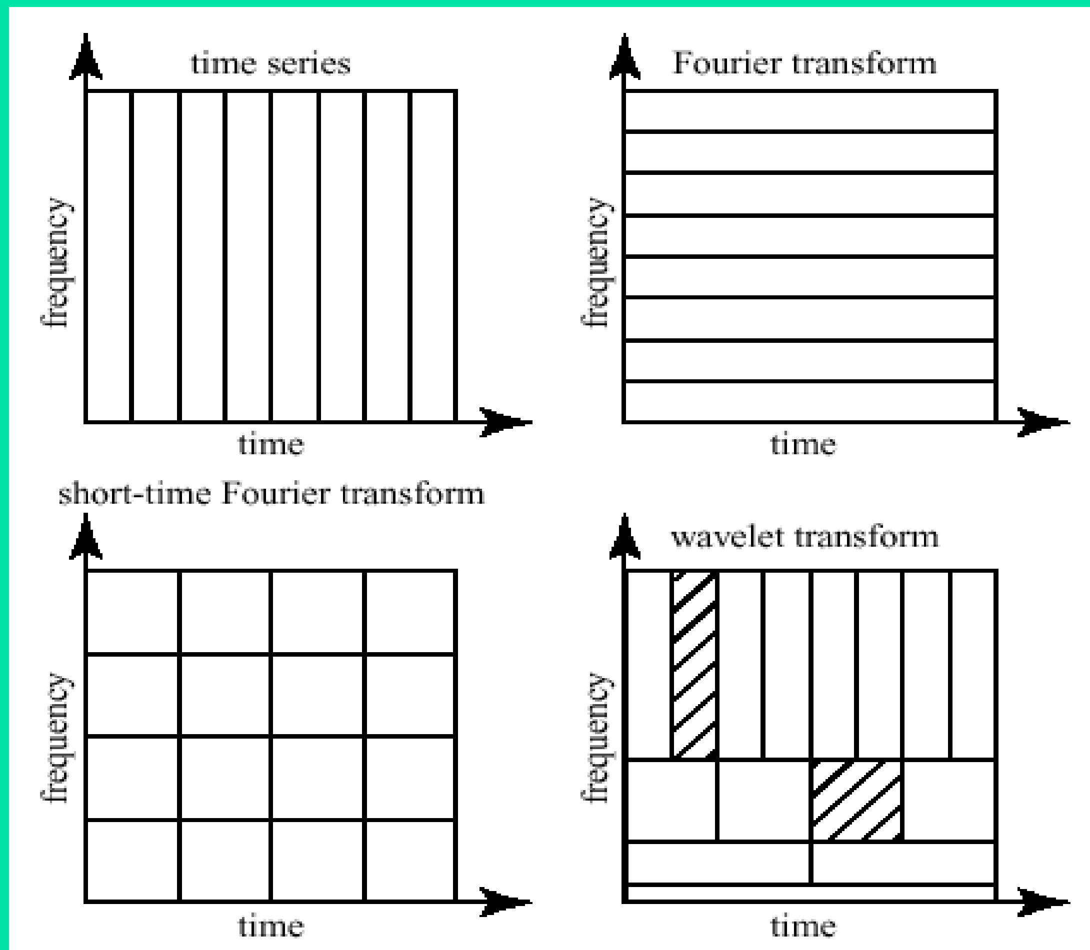
Step 7: CWT is obtained if all s are calculated.

RESOLUTION OF TIME & FREQUENCY



- Each box represents a equal portion
- Resolution in STFT is selected once for entire analysis

COMPARSION OF TRANSFORMATIONS



From http://www.cerm.unifi.it/EUcourse2001/Gunther_lecturenotes.pdf, p.10

DISCRETIZATION OF CWT

- It is Necessary to Sample the Time-Frequency (scale) Plane.
- At High Scale s (Lower Frequency f), the Sampling Rate N can be Decreased.
- The Scale Parameter s is Normally Discretized on a Logarithmic Grid.
- The most Common Value is 2.
- The Discretized CWT is not a True Discrete Transform

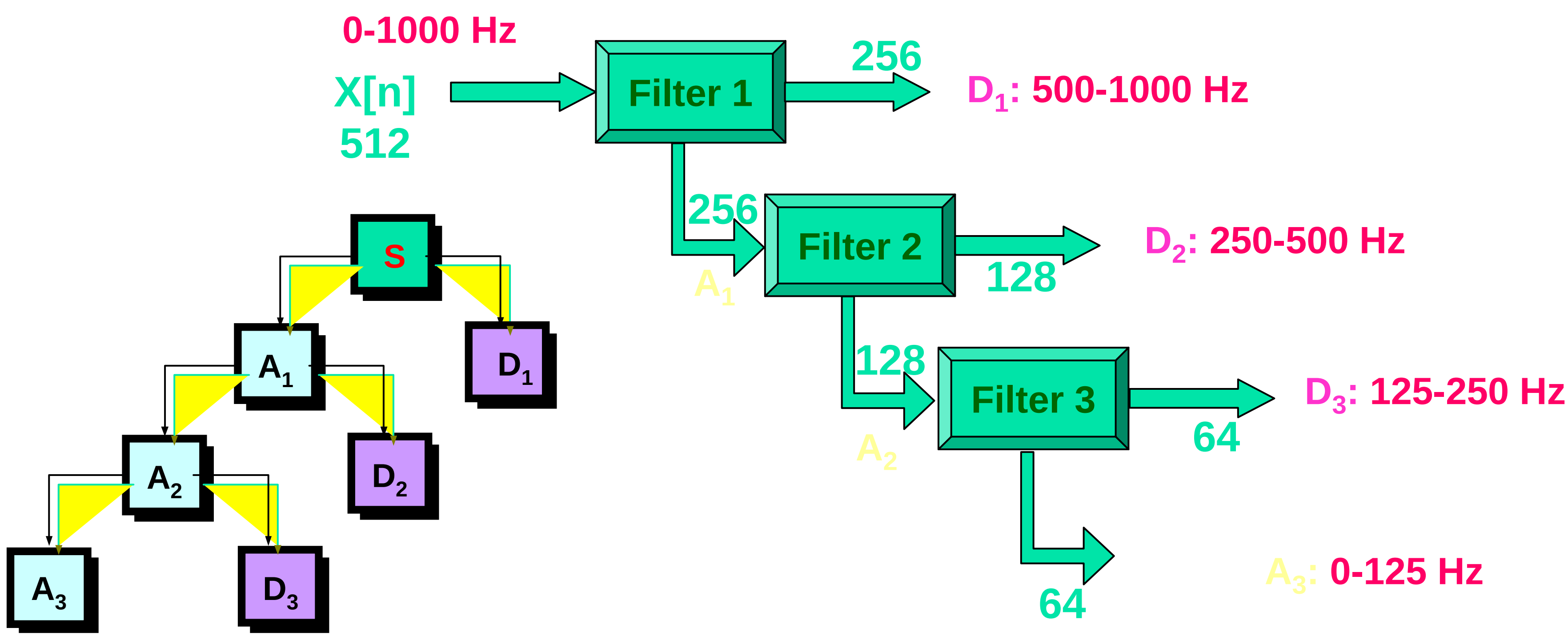
- Discrete Wavelet Transform (DWT)
  - Provides sufficient information both for analysis and synthesis
  - Reduce the computation time sufficiently
  - Easier to implement
  - Analyze the signal at different frequency bands with different resolutions
  - Decompose the signal into a coarse approximation and detail information

Multi Resolution Analysis

- Analyzing a signal both in time domain and frequency domain is needed many a times
  - But resolutions in both domains is limited by Heisenberg uncertainty principle
- Analysis (MRA) overcomes this , how?
  - Gives good time resolution and poor frequency resolution at high frequencies and good frequency resolution and poor time resolution at low frequencies
  - This helps as most natural signals have low frequency content spread over long duration and high frequency content for short durations

SUBBABD CODING ALGORITHM

- **Halves the Time Resolution**
  - **Only half number of samples resulted**
- **Doubles the Frequency Resolution**
  - **The spanned frequency band halved**



RECONSTRUCTION

- What
  - How those components can be assembled back into the original signal without loss of information?
  - A Process After *decomposition* or *analysis*.
  - Also called *synthesis*
- How
  - Reconstruct the signal from the wavelet coefficients
  - Where wavelet analysis involves filtering and down sampling, the wavelet reconstruction process consists of up sampling and filtering

WAVELET APPLICATIONS

- Typical Application Fields
  - Astronomy, acoustics, nuclear engineering, sub-band coding, signal and image processing, neurophysiology, music, magnetic resonance imaging, speech discrimination, optics, fractals, turbulence, earthquake-prediction, radar, human vision, and pure mathematics applications
- Sample Applications
  - Identifying pure frequencies
  - De-noising signals
  - Detecting discontinuities and breakdown points
  - Detecting self-similarity
  - Compressing images

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Thank You